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ABSTRACT

This project had three major goals: (1) investigate the extent to which introductory physics students misuse or misunderstand formulas; (2) catalogue the typical ways in which they do this; and (3) begin the larger task of identifying key types of knowledge that successful problem solvers use to give formulas meaning. Exploratory interviews and group sampling studies were conducted. The interviews were conducted with approximately 25 freshmen and sophomore engineering students. As a result, the project was able to discover new misconceptions about qualitative concepts in physics, develop and refine more simple and elegant problems which would expose and isolate those misconceptions with a minimum of distraction from other possible difficulties, and form hypotheses about four levels of knowledge being used in successful problem solving. A series of three different 45-minute diagnostic tests were conducted with entering freshman engineering majors, using sample sizes of 150, 34, and 38 respectively. These each involved approximately 18 of the questions which had been pilot tested in interviews, including both algebra and physics questions. A parallel test was given to an older group of 24 engineering majors who had just completed a course in introductory mechanics. These tests allowed for the comparison of students before and after taking introductory physics to determine whether the students' learning had been formula-centered. Findings and comments on the research methodology are presented in this final report. (Author)

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FINAL REPORT:

IDENTIFYING DIFFERENT LEVELS OF UNDERSTANDING ATTAINED BY PHYSICS STUDENTS

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John Clement
Research Director

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Identifying Different Levels of Understanding Attained by Physics Students

PART II–SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

Objective: This project investigated the ways in which students taking physics courses at the introductory college level misunderstand or misuse formulas.
Major Findings: (1) Many students can take an overly "formula-centered" approach to learning physics in which they use memorized formulas with little understanding of their meaning. By using clinical interviews and written tests we have isolated two major aspects of difficulty: (A) misconceptions about qualitative concepts in physics: the student can combine and manipulate formulas algebraically but lacks a qualitative understanding of the physical situation; (B)the student can combine and manipulate formulas algebraically but cannot translate between equations and other symbol systems such as data tables, verbal descriptions, or diagrams. It is important for teachers to be able to separate the difficulties described in (A) and (B) above because the remedial teaching strategies in each case are quite different. (2) In area (A), qualitative misconceptions, a catalogue of common preconceptions exhibited by beginning freshmen students has been expanded. (3) Data collected using group tests with 150 students indicates that many of the misconceptions in the catalogue are widespread; they are present in 20-80% of freshman engineers (depending on the particular misconception). (4) In area (B), translation difficulties, it was discovered that even relatively advanced students have severe difficulty in symbolizing certain relationships via algebraic equations. (5) The most common error is a reversal error, where a factor of proportionality is placed on the wrong side of an equation. Isolation of the reversal error represents a breakthrough to us because it exposes a new domain of hidden difficulties surrounding the concept of variable. (6) A first-order theory of understanding in physics was developed which states that a necessary component of understanding involves a knowledge of not one, but four basic kinds of knowledge domains. (7) The finding that understanding physics involves the ability to translate between different systems of representation has had a significant impact on the development of an experimental physics course. Over 50% of the problems developed for the course emphasize translation explicitly.

In summary, we have found that students' hidden misconceptions can be isolated by using carefully designed questions and clinical interviews.

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INTRODUCTION

This project investigated the ways in which students taking physics courses at the introductory college level misunderstand or misuse formulas. Although memorizing formulas and symbol manipulation algorithms is a method most students must use in learning physics, when this is the only type of knowledge that the student has, certain serious problems can arise. For example, the student may know a procedure for calculating the acceleration of an object, given its velocity function. But the same student may do surprisingly poorly when asked to give a verbal or graphical description of an everyday occurrence (such as riding a bike over a hill) in terms of the concept of acceleration. (See the transcript in Appendix I for a detailed example.) In this case the student can derive the formula for predicting acceleration, but his or her understanding of the underlying concept of acceleration is weak. The student has a procedure for "getting the right answer" in special cases, but demonstrates little understanding of the concept when asked to apply it to a practical situation. We describe such a student as having a "formula-centered" approach to the subject.

Goals. Three major goals of the project were to:

(1) investigate the extent to which introductory physics students misuse or misunderstand formulas;

(2) catalogue the typical ways in which they do this; and

(3) begin the larger task of identifying key types of knowledge that successful problem solvers use to give formulas meaning.

In summary, we felt this study would be a step in interesting faculty in the process of increasing the level of understanding their students attain as opposed to merely increasing formula-shifting competence.

BACKGROUND OF THE STUDY

In a preceding one-year RULE grant, we began a catalogue of intuitive physical conceptions of engineering students taking introductory physics. In part of this RISE project, we continued and expanded our study of the students' qualitative intuitions for concepts such as momentum, force, energy, and Newton's three laws. The exploratory studies that had proved to be most startling to other faculty members in the department are those that show the extent to which some physics students acquire only a superficial knowledge of formulas from physics courses and, indeed, the extent to which the very process of being able to symbolize practical physical relationships in terms of a mathematical statement is foreign to many students. The possibility of documenting these findings in a systematic manner by means of clinical interviewing and group testing techniques provided a motive for the present grant. There is some continuity between the present study and previous studies. For example, Erlwanger (1974) in a study of elementary school mathematics students, was able to demonstrate a large gap between the students' own intuitive conceptions of quantitative relationships and the students' knowledge structures in the form of symbol manipulation rules used to cope with their daily assignments. This gap suggests that students did "school math" inside of school and "intuitive math" outside of school with little useful transfer between these two knowledge domains in either direction.

We have found that a similar gap exists between "school physics" and "intuitive physics".

DATA COLLECTION ACTIVITIES

Overall, the project succeeded in identifying two major categories of "formula centered" approaches: (A) misconceptions about qualitative concepts in physics: the student can combine and manipulate formulas algebraically but lacks a qualitative understanding of the physical situation; (B) the student can combine and manipulate formulas algebraically but cannot translate between equations and other symbol systems such as data tables, verbal descriptions, or diagrams. It is important for teachers to be able to separate the difficulties described in (A) and (B) above because the remedial teaching strategies in each case are quite different.

Two types of research were conducted: exploratory interviews and group sampling studies. In the exploratory interviews we sought to: 1) idendify different types of limitations associated with a formula-centered approach; and 2) indentify new and previously unanalyzed phenomena in natural problem solving behavior related to types of knowledge used by successful problem solvers. Following these exploratory studies we developed standard questions for assessing the degree to which the difficulty is widespread.

Exploratory Interviews. We should emphasize that exploratory interviews played a major role in the success of this project. They were conducted most heavily at the beginning of the project but continued throughout and were conducted with approximately 25 freshman and sophomore engineering students. They allowed us to:

(1) discover new misconceptions;

(2) develop and refine simpler, more elegant problems which would expose and isolate those misconceptions with a minimum of distraction from other possible difficulties (such as the Coin Problem, Appendix II);

(3) form hypotheses about four levels of knowledge being used in successful problem solving (see Appendix V).

Diagnostic Tests A series of three different 45 minute diagnostic tests were conducted with entering freshman engineering majors, using sample sizes of 150, 34, and 38, respectively. These each involved approximately 18 of the questions which had been pilot tested in interviews, including both physics and algebra questions. This provided us with a large data base on the entry behavior of engineers. We also gave a parallel test to an older group of 24 engineering majors who had just completed a course in introductory mechanics. This allowed us to compare the performance of students before and after taking introductory physics, in order to determine whether the students' learning had been "formula centered". (The post course results were actually analyzed later under the subsequent NIE-NSF funded project, using an expanded sample of 43 students.)

SPECIFIC FINDINGS

The above activities produced findings in the following three areas:

(1) Misuse of Formulas in Introductory Mechanics. Many physics students show formula centered tendencies when they use formulas with little

understanding of their meaning. Clinical interviews were used to find situations where this occurs, and problems were then refined for use in group tests. For example, the following two problems were given after the relevant course instruction:

> Two ping-pong balls are near each other on a smooth table. One of the balls has a charge q and the other a charge 4q of the same polarity. Describe in words the momentum of each of the ping-pong balls one second after they start to move away from each other because of the Coulomb repulsion.

Two-hundred-and-fifty-nine students took the exam that included this problem. 48% of those answering gave answers that were inconsistent with the physicist's point of view. Many indicated that the force on the ball with smaller charge would be greater while others confounded the concepts of mass and charge and indicated that this ball would be easier to accelerate. However, on an earlier problem where they had to employ Coulomb's law to calculate the force on a point charge, 95% of the students answered correctly. In other words, the students could perform calculations using the formula but when asked to give a coherent verbal description of a situation, widespread misconceptions suddenly appeared.

In considering a second example, we note that another symptom of formula-centered knowledge is the tendency to use a formula in an inappropriate context. This is particularly likely to occur in problems containing extra information. An example of such a problem is the following:

> Slingshot problem. A 100g. projectile is placed in a slingshot and the band is pulled back 0.5 meters and held with a force of 50 newtons before being released. The slingshot takes .05 seconds to accelerate the projectile to its final speed. What is its final speed?

This problem can be solved by recognizing that the potential energy stored in the slingshot ($U_{spring} = \frac{1}{2}kx^2$) will be converted into the kinetic energy of the projectile ($E_k = \frac{1}{2}mv^2$). (Students were given these relations on the cover of the test.) The elasticity constant, k, can be computed from the ratio of the holding force and the distance the band is stretched. However, 71% of the students obtained a solution using the equation $V = at$, an equation which is inappropriate for this problem since the acceleration is not constant. These students calculate a value for acceleration from the equation $F = ma$ by plugging in values for the initial force and mass of the object.

The use of an inappropriate equation signals that students are relying on matching variables given in the problem to variables in standard formulas instead of first thinking through the problem adequately in terms of qualitative physics. By qualitative physics, we mean arguments of the form: "as the elastic band unstretches, it will exert less force and therefore cause less acceleration. $V = at$ describes a constant acceleration producing a constant increase in velocity so that equation can't apply here." The knowledge used in this type of argument is an essential part of competency in physics. This kind of knowledge is not acquired by simply memorizing a formula but rather involves developing a conceptual understanding of the kind of qualitative situation to which the formula applies. In other words, the formula itself does not carry the information that tells one when to use it.

These results show that it is all too easy for us to assume that when a student uses a formula successfully to calculate an answer, he must understand the conceptual model behind it. Our examples show that this can be far from the case. Three types of questions that are particularly useful for exposing the degree of conceptual understanding possessed by students are: drawing qualitative graphs; giving a coherent verbal description of events in an experiment; and solving problems with extra information that can trigger the use of an inappropriate formula. These findings are summarized in Appendix I , "Limitations of Formula-Centered Approaches to Problem Solving in Physics and Engineering."

In a second part of the study of qualitative misconceptions in physics, a catalogue of common preconceptions exhibited by beginning freshmen students was expanded. Data collected using group tests with 150 students indicates that many of the misconceptions in the catalogue are widespread; they are present in 20-80% of freshman engineers (depending on the particular misconception). Again, interviews were used to refine and develop questions for use on written tests. For example, the Coin problem discussed in Appendix II was developed. We found that many physics students have stable, alternative views of key concepts such as elastic forces, momentum, and the relationship between force and acceleration. These "conceptual primitives" are misunderstood at the qualitative level in addition to any difficulties that might occur with mathematical formulation. The source of these qualitative misunderstandings can often be traced to deep-seated preconceptions held by students, which make the comprehension of new concepts in the classroom very difficult. An important implication for instruction is that in the presence of preconceptions learning becomes a process in which new concepts must displace stable concepts that the student has constructed over many years. Under these conditions, teaching strategies limited to expository presentation are unlikely to succeed.

The general theoretical implication of these findings is that although various general reasoning skills are important in physics, domain-specific knowledge is also crucial. Knowledge structures which represent specific types of physical interactions must be structured in a particular way if they are to embody Newtonian concepts; but the alternative knowledge structures found in students often imply very different concepts. This leads to another important implication for instruction. When students with these alternative knowledge structures produce incorrect answers in the classroom, the instructor may in many cases assume that the cause is poorly-developed reasoning skills when in fact the cause is the stability of the student's alternative knowledge structure. It is important for teachers to become sensitive to such distinctions because the indicated teaching strategies are quite different in each case. The following list summarizes the major findings in this area:

(1) Eight major preconceptions about physical phenomena held by students before they take physics were catalogued. A typical example is the widely held belief that a table supporting an orange is not pushing up on the orange, but is merely "in the way" of the orange's fall. This belief is apparently a symptom of viewing forces anthropomorphically as originating only from active sources of power. Another pervasive preconception is that motion of an object implies that continuing presence of a force to cause the motion. If they are dealt with at all, these qualitative issues are usually glossed over briefly in introductory courses and receive only a fraction of the attention they require.

(2) We now have evidence that many of these preconceptions are widespread - that is, many misconceptions are evident in 20% to 80% of our students, depending on the particular misconception.

(3) From tests given to students after they have taken physics courses, we now have evidence that certain qualitative preconceptions are highly resilient (resistent to change) in standard physics courses. For example, the "force of the hand" misconception for the coin toss

problem described in Appendix I is evident in 65% of the students graduating from a calculus based physics course.

These findings are summarized in the following paper: Appendix II "The Importance of Preconceptions and Misconceptions in Introductory Mechanics" (completed under our NSF-NIE grant and presented at the national meeting of the American Association of Physics Teachers, January, 1980.)

(2) Translating from Words to Equations. A second aspect of formula-centered learning emerged with the discovery that even relatively advanced students have difficulty in symbolizing certain relationships via algebraic equations. The most common error is a reversal error, where a factor of proportionality is placed on the wrong side of an equation. Isolation of the reversal error represents a breakthrough to us because it exposes a new domain of hidden difficulties surrounding the concept of variable. By using a cycle of exploratory interviews and question reformulation, we refined several questions which focus on these difficulties, and then conducted group tests with the questions.

Figure 1 shows three problems which ask students to translate various kinds of information into algebraic equations. We have been surprised and disturbed by our results: on problems 1 and 2, 53% and 73% of the subjects majoring in engineering were unable to perform these simple translations. Our first task was to understand how it is possible for such a large proportion of the science oriented college population to fail such simple problems. Two findings are relevant here: (1) The most common error in these problems is the reversal error,

1. Weights are hung on the end of a spring and the stretch of the spring is measured. The data are shown in the table below.

Stretch S (cm)	Weight W (g)
3	100
6	200
9	300
12	400

n=34

percent incorrect: 53%

Write an equation that will allow you to predict the stretch (S) given the weight (W).

2. Write an equation using the variables C and S to represent the following statement:

"At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered streudel."

Let C represent the number of cheesecakes ordered and let S represent the number of streudels ordered.

n=150

percent incorrect: 73%

3. Write an equation using the variables S and P to represent the following statement:

"There are six times as many students as professors at this universtiy."

Use S for the number of students and P for the number of professors.

n=150

percent incorrect: 37%

Figure 1

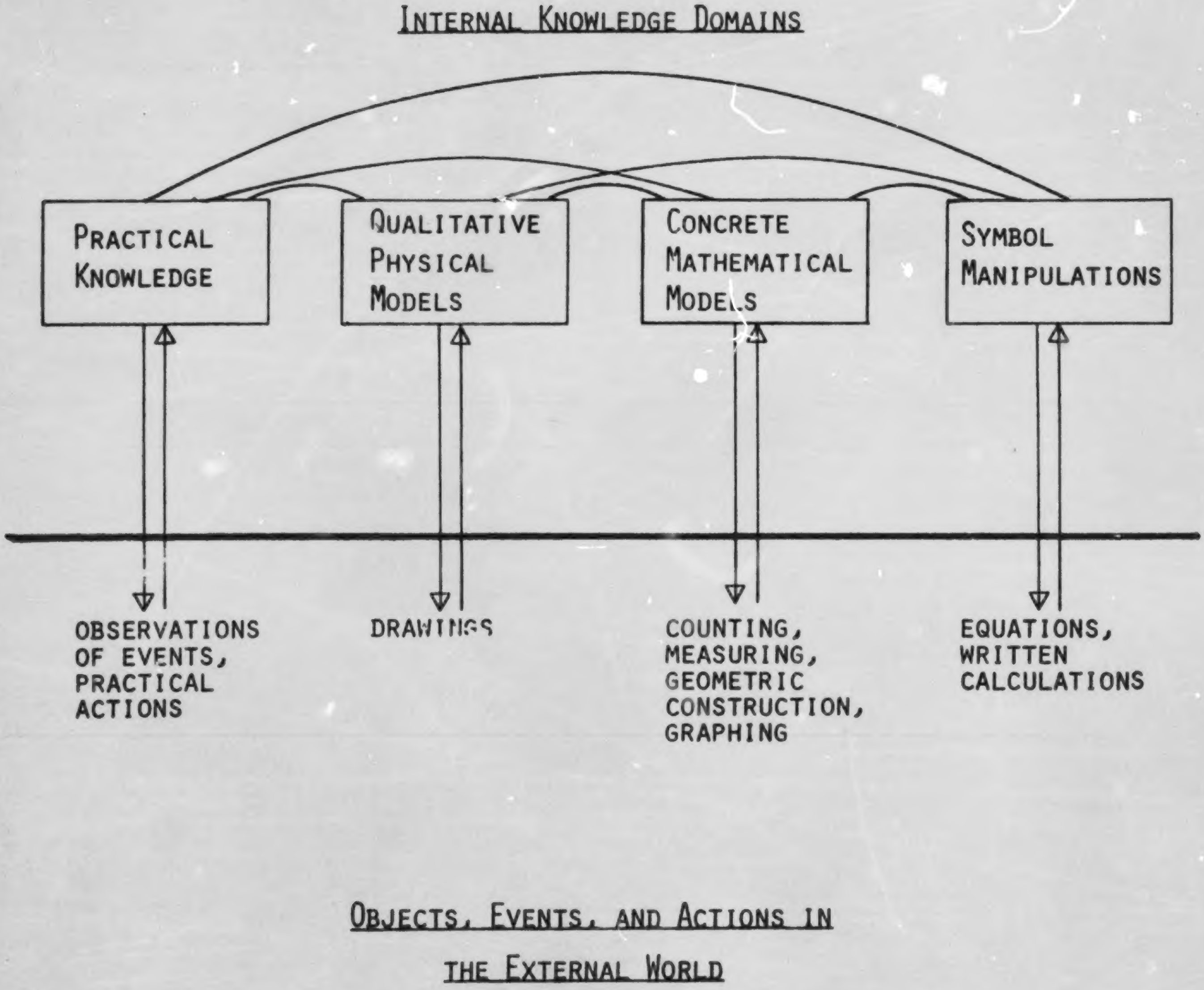
where, in problem 3 for example, the student writes $6S = P$ instead of $S = 6P$. We have proposed a hypothesis for the source of this kind of error, which reflects on the student's concept of variable and function (see Appendix III).

(2) At first, we thought the reversal error might be due to "tricky wording" of the questions or to carelessness on the part of students, but we have become convinced for several reasons that the reversals represent a real cognitive difficulty. In subsequent studies we modified the problems to test the robustness of the effect. The finding has turned out to be surprisingly robust across different populations, and across different types of problems. Large numbers of reversals have been observed in problems involving translations not only from English to equations, but also from equations to English, data tables to equations, and pictures to equations, and in problems involving additive functions as well as multiplicative ones.

It appears that these students have developed special purpose translation algorithms which work for many textbook problems, but which cannot reasonably be called a semantic understanding of algebra. These findings are discussed in "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems", Appendix III ; and "Solving Algebra Word Problems: An Analysis of a Clinical Interview", Appendix IV.

(3) Beyond Formula Centered Learning: Types of Knowledge Needed to Understand Physics. As we studied the difficulties encountered by formula centered students, we were able to begin to identify several of the key abilities possessed by students who do achieve a high level of understanding. We have developed a first-order theory of understanding in physics which states that a necessary component of understanding involves a knowledge of not one, but four basic kinds of knowledge domains or representational systems. These systems are summarized in figure 2 and the theory is elaborated in Appendix V "Some Types of Knowledge

Fig. 2 Representational Systems
(Knowledge Domains) Employed in Physics


INTERNAL KNOWLEDGE DOMAINS
PRACTICAL KNOWLEDGE
QUALITATIVE PHYSICAL MODELS
CONCRETE MATHEMATICAL MODELS
SYMBOL MANIPULATIONS
OBSERVATIONS OF EVENTS, PRACTICAL ACTIONS
DRAWINGS
COUNTING, MEASURING, GEOMETRIC CONSTRUCTION, GRAPHING
EQUATIONS, WRITTEN CALCULATIONS
OBJECTS, EVENTS, AND ACTIONS IN THE EXTERNAL WORLD

Used in Understanding Physics." Our general hypothesis is that a usable knowledge of physics formulas must be based on a knowledge of mathematical models and a knowledge of qualitative physical models, and should ultimately be connected to knowledge of practical experiences. The student must be able to describe observed physical events, describe the behavior of physical models (such as the kinetic model of gasses), work with graphs and mathematical diagrams, and manipulate equations according to the rules of algebra and calculus.

This theory of four types of knowledge used in physics allows us to recast the question of whether the formal exposition of physics in terms of formulas is sufficient for what the physics student needs to learn; one can see that formal expositions emphasize heavily the use of written formulas in the symbol manipulations domain. The danger here is that a student may get "stuck" in the symbol manipulation mode--he may learn a certain set of equations, but not understand their meaningful interpretation in the form of physical models, mathematical models, or practical actions. Making sure that these connections are made is a worthwhile goal and a real pedagogical challenge. An "Outline of Observable Indicators of Understanding" was also developed and is included in Appendix VI.

(4) Difficulties in Our Research on the Concept of Acceleration. We have had to put aside one of our lines of research for the moment because of an interesting methodological problem. The problem arose in a study of the concept of acceleration where students were asked to draw acceleration versus time graphs for simple motions. Although we uncovered a large

number of interesting errors on these tasks, we found in the protocol analyses that we could not discriminate well between misconceptions related to the concept of acceleration and certain misconceptions related to graphing skills which we were already aware of. This experience has heightened our awareness of the importance of discriminating between two major types of misconceptions: those concerning conceptual primitives in the subject area and those concerning symbolization skills such as graphing. This has affected our methods of question formulation in other areas. Further progress will require the development of more specific problems in each area.

EFFECT OF THE RESEARCH ON AN EXPERIMENTAL COURSE

We have found many of our findings to have immediate implications for instruction. The finding that understanding physics involves the ability to translate between different systems of representation has had a significant impact on an experimental physics course that has been developed by our FIPSE project. In fact, it has become a unifying theme for the freshmen engineering course, Introduction to Analytical Techniques for Physics. This experimental course is a pre-physics course designed to develop problem solving skills necessary for freshman engineers to be more successful in physics. The course emphasizes translation skills between verbal descriptions, equations, graphs, pictures, data tables, and vector notations. Students were also asked to invent standard physics formulas from qualitative descriptions of physical law. The primary instructional activity in this course was problem solving with help from peers and instructors. Over 50% of the problems developed for the course now emphasize translation explicitly (see Appendix III for examples).

In addition, a laboratory which emphasizes qualitative physics has been developed for the course. (Developed with both FIPSE and RISE support). This laboratory utilizes similar types of translation questions starting from qualitative (and only later, quantitative) observations. (See Appendix VII).

COMMENTS ON CRITICAL FEATURES OF THE METHODOLOGY USED IN THIS STUDY

Role of the Exploratory Interview. A distinctive feature of this type of study is its strong emphasis on the initial use of exploratory clinical interviews. When these protocols involve relatively spontaneous behavior, they can contain real surprises, and they can provide a clear basis for indentifying new species of knowledge structures. This contrasts with cases where researchers have done considerable theory development before tying the theory carefully to the behavior of human subjects; in that case there is a natural tendency to choose tasks which elicit behavior compatible with the existing theory. Thus, research which develops new tasks from clues in previous protocols in an open-ended way may be particularly valuable in providing new behavior patterns that have not previously been observed.

In such exploratory interviews a preliminary picture of the typical characteristics of the sample as a whole will emerge, but since very little is known about the structure of particular intuitive conceptions, detailed clinical observations and theory construction techniques--leading to the identification of new factors--comprise the appropriate methodology. We prefer this method to statistical comparisons of previously identified factors, as the starting point of a study. Large sample size is not a crucial factor in this phase, since the cognitive hypotheses proposed are to be evaluated initially on the basis of how well they account for detailed sequences of behavior in individual students. What is crucial is that (1) questions are discovered which expose misconceptions, and (2) that subjects be interviewed in an atmosphere which encourages them to think spontaneously and naturally, and that encourages them to verbalize their solution methods in an uninhibited manner. Analysis of such micro-case studies is especially helpful in forming new hypotheses concerning the nature of particular systems of preconceptions and representational transformations.

Group Tests. We have so far used group testing primarily as a method for demonstrating how widespread particular misconceptions are. This can be done with validity only after (1) a new difficulty is discovered in interviews or observations of schoolwork, (2) the nature of the misconception has been "pinned down" in exploratory interviews, and (3) questions have been refined specifically for use in group tests which are free from interference by other possible sources of error. The interaction of hypotheses, interview data, and group test data in our methodology described above is shown in Figure 3.

We plan to use group tests in a more experimental way in the future, but we believe that clinical interviews are the best starting point for

METHODOLOGY

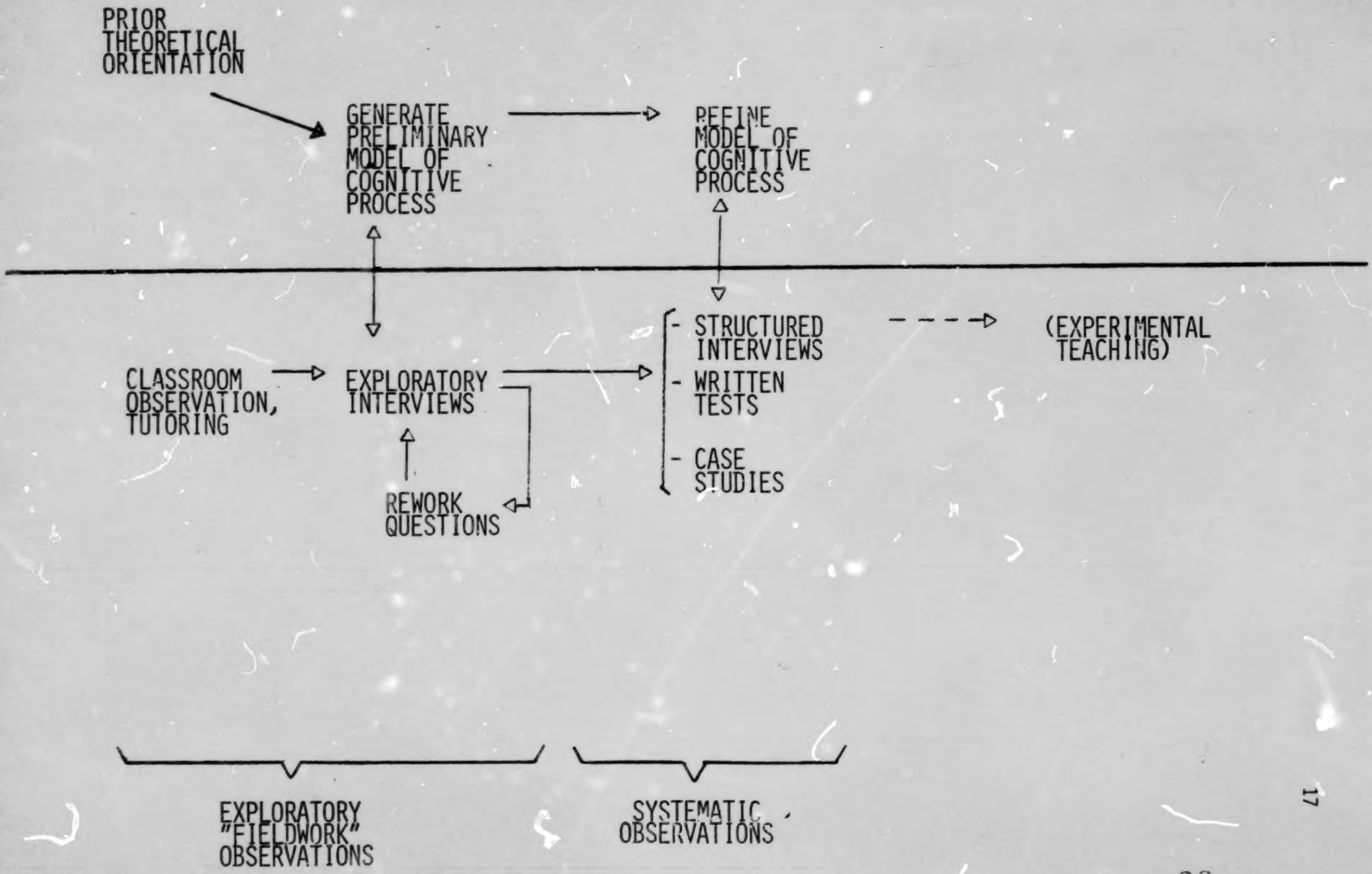

PRIOR THEORETICAL ORIENTATION
GENERATE PRELIMINARY MODEL OF COGNITIVE PROCESS
REFINE MODEL OF COGNITIVE PROCESS
CLASSROOM OBSERVATION, TUTORING
EXPLORATORY INTERVIEWS
REWORK QUESTIONS
- STRUCTURED INTERVIEWS
- WRITTEN TESTS
- CASE STUDIES
(EXPERIMENTAL TEACHING)
EXPLORATORY "FIELDWORK" OBSERVATIONS
SYSTEMATIC OBSERVATIONS

Figure 3

producing cognitive hypotheses that are anchored in observations of naturalistic behavior.

Protocol Analysis. The analysis of protocols can be broken down into several steps:

1. Conduct interview
2. Transcribe protocol
3. Identify phenomena and behavior patterns in the tape with the help of the typed protocol
4. Interpret phenomena in terms of cognitive models
5. Continue alternating between steps 3 and 4 in order to refine the model

The methodology of protocol analysis has been discussed by Witz and Easley (forthcoming). Cronbach has reversed his methodological preferences in educational research in recent years more and more toward the clinical approach. Easley (1977) quotes Cronbach as saying:

> Instead of making generalization the ruling consideration in our research, I suggest that we reverse our priorities. An observer collecting data in one particular situation is in a position to apprraise a practice or proposition in that setting, observing effects in context. In trying to describe and account for what happened, he will give attention to whatever variables were controlled, but he will give equally careful attention to uncontrolled conditions, to personal characteristics, and to events that occurred during treatment and measurement. As he goes from situation to situation, his first task is to describe and interpret the effect anew in each locale, perhaps taking into account factors unique to that locale or series of events.[1]

As the analyst goes from protocol to protocol he must constantly be on the lookout for new phenomena, and in interpreting the phenomena it is always necessary to pay attention to the immediately previous series of events, which may be unique to that particular protocol. For these purposes the video tape recorder plays a role that is analogous to that of the microscope in biology. Viewing repeated playbacks of key tape sections allows one to make much more detailed inferences concerning internal processes in the student.

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[1] "Monograph on Clinical Studies in Mathematics Education." (Commissioned by ERIC.) Information Reference Centers for Science, Mathematics, and Environmental Education, Ohio State University, 1977, p. 4.

Levels of Data Analysis. Data analysis has taken place at three different levels. At the first level, student responses that are non-trivial "errors" from the expert's point of view are collected and catalogued. These detailed descriptions of typical error sequences by themselves can provide valuable feedback to teachers. Reports at this level include transcript exerpts to provide readers with examples of "live" problem solving behavior. At the second level analysis of patterns in these errors are made to produce models of or determine characteristics of the knowledge structures which cause them. The aim is to provide an analysis of the student's internal "initial state" before courses and subsequent states during or after courses. These analyses will in turn focus attention on implicit but necessary structures, lacking in students but present in experts. (Appendix II gives an example of the first level of analysis. Page 8 of the same Appendix is an example of the second level of analysis, as is Appendix III).

Some preconceptions and misconceptions are more difficult to decipher than others. A third level of analysis examines protocols or key protocol sections for rigorous analysis at a finer level of detail. This level is appropriate for addressing more difficult research questions such as describing the detailed characteristics of causal knowledge structures and of different representational systems and their interactions during problem solving. (Appendix IV gives an example of a paper written at this third level.)

Summary of Research Approach. Our approach to research is similar to that of ethology in its empirical emphasis. We feel that it is important at this stage in the development of cognitive science to develop a broader empirical base by recording careful observations of the types of problem solving behavior that humans are capable of, regardless of whether such behavior fits existing theories of problem solving. Second, analysis of such data can lead to qualitative first order models of the internal cognitive structures and processes involved, and these can be shared, argued over, and improved by scientists even though detailed explanations of the "atoms" used in the model are not yet fully developed. We suspect that such first order models of complex behavior can be of much more use to educators than highly detailed models of trivial behavior.

Understanding human knowledge structures and the process of translating between representations is an ambitious goal. The processes are complex and their investigation requires the time-consuming method of the clinical interview. Progress can be made now, but solutions to the problems are a long-range goal. At the same time, the goal of producing catalogues and first-level analyses of common misconceptions is immediately attainable and has clear benefits for education: it can provide educators with information about their students' acquired attitudes and mental habits, and the possible consequences of such mental structures on the materials of instruction. We have found that it is possible to work on these long and short range goals together, and that the two pursuits are complementary. Contact with ongoing courses where real instructional problems arise daily is a definite asset to a research program in this area because new and interesting misconceptions crop up that would not have been discovered in the more formal parts of the study. Such contact also strengthens our focus on research questions that have meaningful educational implications. In sum, we feel that these are the strengths of this research approach:

--parallel investigations via clinical interview studies and group testing

--a focus on critical long-range problems, i.e., the role of implicit knowledge and translations between representations

--short range focus on a catalogue of preconceptions and misconceptions which is both an empirical base for research and a resource for educators

--continuing contact with courses and instructional problems that arise under "real world" conditions.

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LIMITATIONS OF FORMULA-CENTERED APPROACHES TO PROBLEM SOLVING IN PHYSICS AND ENGINEERING*

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ABSTRACT

Transcripts of two freshmen engineering majors solving elementary physics problems are presented in order to examine some of the limitations of formula-centered approaches to problem solving. In both cases the student uses a formula successfully but his qualitative conception of the underlying physical situation is very weak. Results from written tests are also presented which indicate that this phenomenon may be quite widespread.

LIMITATIONS OF FORMULA-CENTERED APPROACHES TO PROBLEM SOLVING IN PHYSICS AND ENGINEERING

John Clement

In order to pass most engineering and physics courses students need to memorize certain formulas and learn equation solving techniques. However, when this is the only type of knowledge that the students have, certain serious problems can arise. They may for example, know a procedure for calculating the acceleration of an object, given its velocity as a function of time, but do surprisingly poorly when asked to give a verbal or graphical description of an everyday occurrence involving acceleration.

In this paper transcripts of two freshmen engineering majors solving elementary physics problems are presented in order to examine some of the limitations of formula-centered approaches to problem solving. In both cases the student uses a formula successfully but his qualitative conception of the underlying physical situation is very weak. Results from written tests are also presented which indicate that this phenomenon may be quite widespread.

EXAMPLES FROM INTERVIEWS

Recent psychological research on the cognitive processes involved in problem solving has made use of interview data where students "think aloud" while working on a problem.[1,2,3] Interview data can also give educators some valuable insights into some of the sources of difficulty students encounter in courses.

In the verbatim transcript shown below an engineering student is asked about the concept of acceleration. Jim successfully differentiates an algebraic expression for the speed of an object to obtain the acceleration as a function of time. However, when asked to draw a qualitative graph for the acceleration of a bicycle going through a valley between two hills, he confounds the concept of acceleration with concepts of speed and distance.*

I = Interviewer

J = Jim

1 I: Here's an expression for the speed of an object travelling on a straight line:
(writes: $S_{(t)} = 5t^2 + 2t$)
Can you write an expression for its acceleration?

2 J: That would be (pause) 10t plus 2.

3 I: And how did you get 10t plus 2?

4 J: Acceleration is the derivative of velocity.

5 I: What would the acceleration be after 2 seconds?

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*This problem is related to a problem reported in Monk (1975).

6 J: (Writes: $a = 10t + 2$)
22 ft. per second per second - I think those are the units.
7 I: You substituted 2 for the t?
8 J: Yeah.

At the time of this interview Jim was taking introductory calculus based physics course. The above sequence suggests that he understands the concept of acceleration well. But further probing by the interviewer reveals some hidden gaps in his knowledge. Jim is shown a picture of a road with a bicycle rider on it. (See fig. 1.) He is told that the cyclist always pushes on the pedals with the same amount of effort and asked to describe the speed of the cyclist qualitatively. He describes the speed by sections as shown in fig. 1. The instructor then draws axes for a graph directly under the drawing of the valley, and the following dialogue ensues.

9 I: Let's do (a graph of) acceleration.
10 J: (Jim constructs the graph shown in fig. 2 piece by piece as described below)
That would be zero from here to here
(Draws segment A-B in fig. 2.)
11 I: Why?
12 J: Because, there was no change in your acceleration, it was constant.
13 I: Would you label that B?
14 J: OK. So acceleration is a change in velocity - so that's

zero because there's no change - the change here (b to c in the original picture) was negative - velocity was negative - so that would go down (draws line under B-C in graph) and acceleration zero (points to c-d in original picture) (draws line C-D below axis)

15 I: So what's happening here (c-d) to acceleration?

16 J: It's constant.

17 I: OK - now what?

18 J: Then I get stuck - uhm - velocity's negative (referring to d-e in picture) so acceleration has to be negative - and I'm already negative so I don't know what to do - I guess I'll go down (draws line under D-E in graph)

19 I: Okay

20 J: And then it's constant again like that (draws line under E-F)

There is a striking contrast here between Jim's work in using equations and his performance in using qualitative graphs and verbal description. In the first section, he successfully differentiates an expression for the speed of an object to find its acceleration. He also gives a correct set of units for acceleration as feet per second per second. In the second section he is able to give a plausible approximate description for how the speed of the bike will change in the valley. But when it comes to his verbal and graphical descriptions of the cyclist's acceleration, serious problems arise. In particular, he describes the acceleration as being constant (but non-zero) when the speed is constant (lines 16 and 20). The interviewer intentionally leaves the choice of the x-axis variable to Jim, but Jim does not see the need to define a choice explicitly. He also seems to confuse the idea of a negative velocity being represented below the axis of a graph with the image of the cyclist dropping into the valley in the picture. And in general he appears to confound the concepts of acceleration and speed in statements such as, (line 18) "Velocity's negative, so acceleration has to be negative."

How can he be so strong in using one means of representation (formulas) and so weak in using others (graphs, verbal description) when for us they refer to the same theoretical concept? It appears to be the case here that even though Jim can use a symbol manipulation algorithm (differentiation of a polynomial) to obtain an algebraic expression for acceleration, his understanding of the underlying concept of acceleration is weak. The student has a procedure for "getting the right answer" in special cases but demonstrates little understanding of the concept when asked to apply it to a practical situation. We describe such a student as having a "formula-centered" view of the concept. No doubt Jim could pass many physics tests using his current formula manipulation skills. But these tests might not reveal his fundamental confusions of the concepts of acceleration and velocity. Qualitative graphs like those included here provide one means of checking up on whether a student's understanding goes beyond a knowledge of formula manipulation techniques. Sketching the graph of acceleration vs. time for a simple pendulum is a similar but even harder problem. These basic problems look very easy but turn out to be surprisingly challenging for students.

Another example of the gap between formula-centered knowledge and conceptual understanding is given in the transcript excerpt below. Here another student is working on the problem of predicting the forces that two positively charged particles will exert on one another. Paul correctly indicates that he can solve for the forces using the same quantitative formula (Coulomb's Law) that a physicist might use. He demonstrates that he can find numerical values for the forces if he is given values for each charge and the distance between the particles. But further probing by the interviewer reveals that his qualitative conception for how the particles affect each other is not the same as the physicist's.

Problem: Given two point charges what are the forces on each?

(Place figure 3 about here.)

(In the context of a previous problem the student was able to write Coulomb's Law from on the blackboard in the form shown)

$$F_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

Transcript excerpts:

1 I: Say this force started out at four, (points to force on q_2) what do you think this force [on q_1] would be?

2 P: It would be equal [i.e., 4].

3 I: [Do] you know why they are equal and opposite?

4 P: Same charge.

5 I: OK. (The interviewer is very surprised by this answer since he expected the student to cite Newton's 3rd Law and say that the forces would always be equal and opposite. But he manages to hide his surprise and design a question to further uncover the the student's model.) Now if I change this charge [q_2] -- increase it to +6 -- what will happen to the forces?

6 P: This one [#1] will be pushed off a lot farther.

7 I: This force [on #1] will go up?

8 P: Yes.

9 I: OK. How about over here [force on #2] --- any change/

10 P: Ah ..... I think that will kind of stay about the same, it'll just push this one [#1] out farther.

11 I: So then if I change this one too? (Changes charge indicated on #1 to +6 as well)

12 P: They [the forces] will both be equal again. But be higher.

13 I: How much higher do you think they will be?

14 P: Ah ..... Just plug it into the formula.

According to this student's conception, forces are equal and opposite when the charges are equal but the forces are unequal when the charges differ. This view conflicts with a basic principle of physics -- that two interacting particles must exert equal and opposite forces on one another (Newton's) Third Law). The student seems to believe that an increase in charge on particle #2 affects only the force on particle #1. Possibly this conception reflects a cognitive preference for a one-directional causal model of the situation rather than an interaction model. But in any case the student's model is qualitatively different than the standard model of the physicist. And whatever knowledge the student has of Newton's Third Law from his previous course in mechanics or from everyday experience has apparently not been accessed appropriately in this situation.

These two protocols illustrate several different aspects of the same general problem. In the case of the first student we find a single weak and undifferentiated concept (acceleration). In the case of the second student we find a conflicting preconception in the form of a non-standard conceptual model (involving several concepts) for how forces are generated between two particles. In the interview this preconception overrides whatever other methods the student might have for analyzing the situation such as starting from the equation or from Newton's Third Law. Both examples show one of the important limitations of formula-centered knowledge. They indicate that a student's qualitative conception of the underlying physical situation can be very weak even when the student can remember and manipulate relevant formulas.

Over the past several years we have collected a large number of interviews of this kind as part of a project on studying the nature of intuitive conceptions and misconceptions in physics. We have found that questions involving qualitative physics concepts are particularly useful for isolating areas where conceptual understanding is weak. Two of the forms these questions can take are: asking students to generate qualitative graphs from verbal descriptions, as in the bicycle problem; asking them to generate verbal descriptions and coherent explanations, as in the repelling charge problem. Such questions can also be used in standard written tests to give information about how wiedspread a certain difficulty is. Examples of such questions are given in the next section.

EXAMPLES OF WRITTEN TEST QUESTIONS

We included the following problem in the mid-term exam of a second semester, calculus-based physics course for engineers (electricity and magnetism):

> Two ping-pong balls are near each other on a smooth table. One of the balls has a charge q and the other a charge 4q of the same polarity. Describe in words the momentum of each of the ping-pong balls one second after they start to move away from each other because of the Coulomb repulsion.*

259 students took the exam and of these, 222 gave an answer for the problem. The results are shown in Table 1. 48% of those answering gave answers that were inconsistent with the physicist's point of view. Many indicated that the force on the ball with smaller charge would

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*The author is indebted to Dr. Jack Lochhead for providing this data.

be greater while others confounded the concepts of mass and charge and indicated that this ball would be easier to accelerate. However, on an earlier problem where they had to employ Coulomb's law to calculate the force on a point charge, 95% of the students answered correctly. Although the first question was labeled "extra credit", the students assumed that the question would count toward their grade. Thus most attempted to answer the question. In summary, the students could perform calculations using the formula but when asked to give a coherent verbal description of a situation, widespread misconceptions suddenly appeared.

In considering a second example, we note that another sympton of formula-centered knowledge is the tendency to use a formula in an inappropriate context. This is particularly likely to occur in problems containing extra information. These problems can confuse students who use a more or less blind strategy of matching variables included in the problem to formulas containing those variables. An example of such a problem is the following:

> Slingshot Problem- A 100 g. projectile is placed in a sling-shot and the band is pulled back 0.5 meters and held with a force of 50 newtons before being released. The slingshot takes .05 seconds to accelerate the projectile to its final speed. What is its final speed?

This problem can be solved by recognizing that the potential energy stored in the slingshot, ($U_{spring} = \frac{1}{2} kx^2$) will be converted into the kinetic energy of the projectile ($E_k = \frac{1}{2} mv^2$). (Students were given these relations on the cover of the test.) The elasticity constant, k, can be computed from the ratio of the holding force and the distance the band is stretched. However many students will obtain a solution using the equation $V = at$, an equation which is inappropriate for this problem since the acceleration is not constant. These students calculate a value for acceleration from the equation $F = ma$ by plugging in values for the initial force and mass of the object.

We gave the slingshot problem to 24 engineering students at the end of their calculus-based mechanics course. The students were paid volunteers from a class of 200 who agreed to take a test just a few days before their final exam. At the end of the term, the average grade in the course for the 24 students was found to be 3.3 as opposed to 2.9 for the entire class. The results in Table 2 show that at least 71% used the inappropriate equation $V = at$ in their solution. The use of an inappropriate equation signals that students are relying on matching variables given in the problem to variables in standard formulas instead of first thinking through the problem adequately in terms of qualitative physics. By qualitative physics, we mean arguments of the form: "as the elastic band unstretches, it will exert less force and therefore cause less acceleration. $V = at$ describes a constant acceleration producing a constant increase in velocity so that equation can't apply here." The knowledge used in this type of argument is an essential part of competency in physics. This kind of knowledge is not aquired by simply memorizing a formula but rather involves developing a conceptual understanding of the kind of qualitative situation to which the formula applies.[2] In other words, the formula itself does not carry the information that tells one when to use it.

CONCLUSION

It is all too easy for us to assume that when a student uses a formula successfully to calculate an answer, he must understand the conceptual model behind it. Our examples show that this can be far from the case. Three types of questions that are particularly useful for exposing the degree of conceptual understanding posessed by students are: drawing qualitative graphs; giving a coherent verbal description of events in an experiment; and solving problems with extra information that can trigger the use of an inappropriate formula.

The students we tested were taught by two experienced, extremely competent instructors who have consistently received high praise for their teaching efforts. They used a modern text and up to date laboratory equipment at a major university in the United States. We believe the conceptual understanding students aquire in a course is more valuable to them in the long run than their ability to remember formulas. We therefore find the above results disturbing. At the same time we find the results fascinating, because first, they are a beginning step in answering the question: "What kinds of knowledge are important for conceptual understanding in physics and engineering?" and secondly, they help us focus in on the particular areas and skills where the most interesting teaching challenges lie.

Acknowledgements

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Electrostatic Problem Results (n = 259)

Correct Answers: Ping Pong Problem	Incorrect Answers:	No Answer	Correct Answers: Coulomb's Law Problem
115	107	37	239
44%	41%	14%	95%

Slingshot Problem Results (n = 24)

Correct Method	Inappropriate Equation v = at	Source of Error Unclear	No Answer
3	17	2	2
13%	71%	8%	8%

Table 1

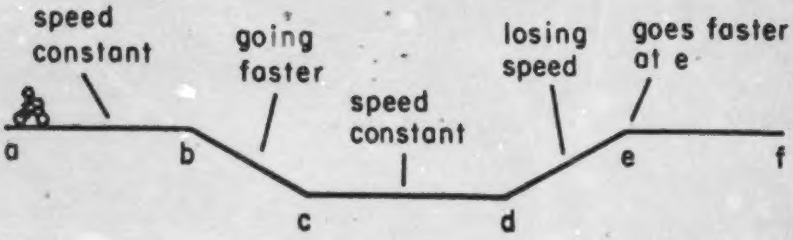
speed constant
going faster
speed constant
losing speed
goes faster at e
a
b
c
d
e
f

Fig. 1

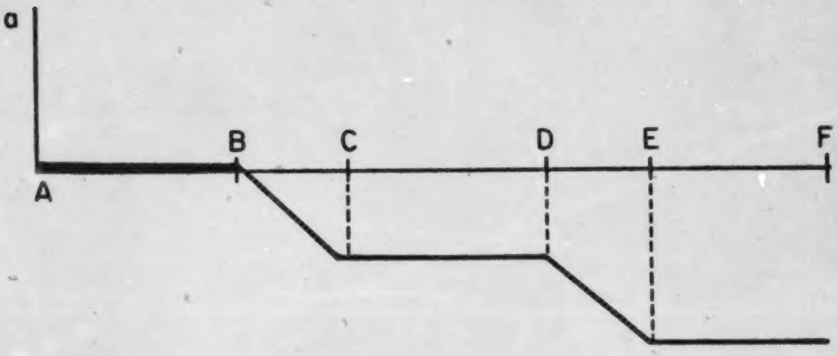

Fig. 2

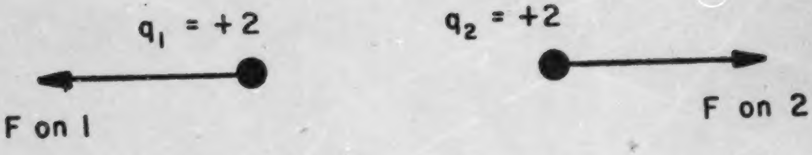
$q_1 = +2$
$q_2 = +2$
F on 1
F on 2

Fig. 3

APPENDIX II

COMMON PRECONCEPTIONS AND MISCONCEPTIONS AS AN IMPORTANT SOURCE OF DIFFICULTY IN PHYSICS COURSES[*]

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July 1979

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Common Preconceptions and Misconceptions as An Important Source of Difficulty in Physics Courses

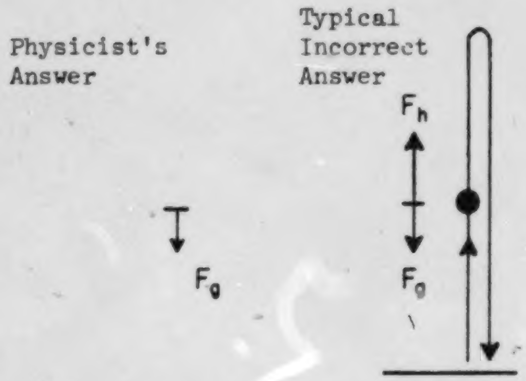
Physics is commonly considered by students to be a difficult subject. When we search for sources of the difficulty that students encounter in physics, we can identify many contributing factors such as abstractness of the material, degree of precision required in problem solving, sophistication in the types of reasoning required including formal reasoning in the Piagetian sense, and mathematical skills required. This paper discusses another source of difficulty that has been widely acknowledged but that has been insufficiently analyzed in the past, namely, the presence of inherently difficult key concepts in physics such as acceleration, momentum, relative motion, charge, potential difference and the relationship between force and acceleration. These concepts are so familiar to practicing physicists that they can forget what it is like to view the world without them, and they can underestimate the learning difficulties they present to the student. We find that many students have difficulty understanding these concepts at the qualitative level, much less at the level of quantitative relationships. These difficulties, however, may go undetected because it often happens that a student's superficial knowledge of formulas and formula manipulation techniques will mask his misunderstanding of underlying qualitative concepts.

This paper gives examples of students' misconceptions in the area of force and motion. This relationship between force and acceleration, summarized in the equation F = ma, appears to be an inherently difficult concept or principle in physics. It seems reasonable that an understanding of F = ma is made difficult because it conflicts with the beginner's intuitive ideas about motion. In the real world, where friction is present, a constant propelling force is implicated as the cause of a constant velocity. This paper provides empirical evidence that many beginners hold this view. In fact, the misconception shows up in a wider diversity of problem situations than one would expect, and especially in the cases where net force opposes the motion. Examples of four situations in which the misconception appears are given based on taped interviews with freshman engineering students. Sample sections of interview transcripts are given on p.13. Furthermore, data comparing performance of students in this area before and after taking a standard, calculus based course indicate that this set of preconceptions is highly resistant to change. It therefore appears to be a major stumbling block in the physics curriculum.

THE "MOTION IMPLIES A FORCE" PRECONCEPTION

The preconception we wish to examine more closely is the belief that whenever one sees motion, there must be a force causing the motion and acting in the same direction as the motion. The following examples of common errors made on qualitative problems were observed during a study in which we interviewed 15 freshman and sophomore engineering majors. They were asked to think aloud as they solved these and other related problems.

Example 1: Forces on the Tossed Coin



A coin is tossed from point A straight up into the air and caught at point E. On the dot to the left of the drawing draw one or more arrows showing the direction of each force acting on the coin when it is at point B. (Draw longer arrows for larger forces).

Typical Student's Answer: While the coin is on the way up, the "force from your hand," F_h, gradually dies away as it pushes up on the coin. On the way up it must be greater then F_g, otherwise the coin would be moving down."

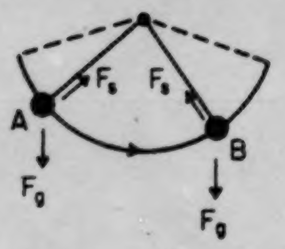
Apparently it is difficult for the student to think of an object continuing to ove in one direction with the total net force acting in the opposite direction. The belief that appears to underlie this response is: "A continued force is necessary for continuing motion in the same direction." We call this the "motion implies a force" misconception. This type of belief shows up in pre-Newtonian theories of motion such as the Aristotelian explanation of the horizontal motion of an arrow after release from the bow via "forward forces from air currents" or an impetus force "injected into" the arrow and travelling with it. What has surprised us is the pervasiveness of this belief and the wide diversity of situations in which it shows up, once one begins to listen to students' common-sense theories. The belief contrasts with the Newtonian view that a net force acts to change the velocity of an object and is not required for a change in the position of an object. We have come to believe however, that students possess strong preconceptions which can prevent this Newtonian view from being assimilated. As an illustration of the diversity of situations in which the motion-implies-a-force belief appears, consider how it enters into the following situations:

Example 2: Forces on a Pendulum

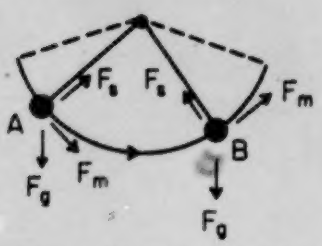
Question: a) A pendulum is swinging from left to right as shown below. Draw arrows showing the direction of each force acting on the pendulum bob at point A. Do not show the total net force and do not include frictional forces. Label each arrow with a name that says what kind of force it is.

b) In a similar way, draw and label arrows showing the direction of each force acting on the pendulum bob when it reaches point B.

Physicist's answer:



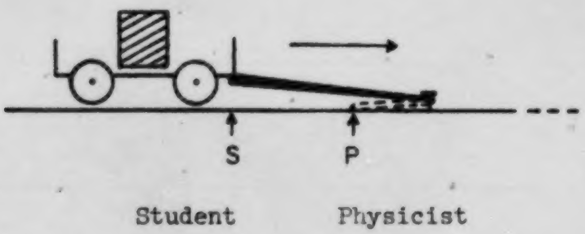
Typical Incorrect Answer:



Typical Incorrect Explanation: F_m is the force that makes the pendulum move. If F_m weren't there, the pendulum could never move up to the top of its swing.

Here, F_m is seen as one of three forces acting on the bob and is seen as the force that "makes the pendulum go up on the other side." F_m is seen as a force that changes direction and as a force neede to explain the changing direction of the motion. Thus, the direction of force is directly coupled to the direction of motion.

Example 3: Launching a Cart -- Velocity Decreases With the Force of the Elastic.



Question: The cart is launched on the table by the elastic band. Where will the cart reach its maximum speed?

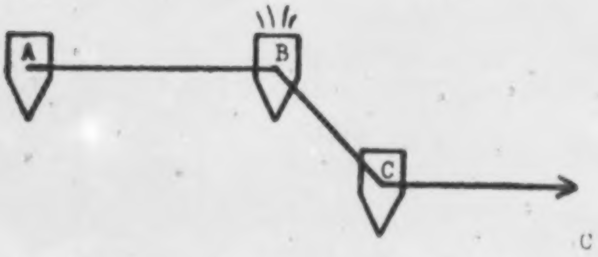
Typical Incorrect Answer: Maximum speed is reached immediately after the cart is released from the hand where the band is stretch the most, because the band is pulling hardest there.

Here, the student feels that there must be a direct correspondence between the quantity of force exerted on the cart and its instantaneous acceleration. Thus the amount of motion is seen to vary directly and immediately with the amount of force.

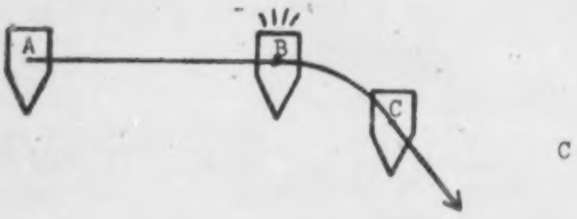
Example 4: a) A rocket is moving along sideways in deep space, with its engine off, from point A to point B. It's not near any planets or other outside forces. Its engine is fired at point B and left on for two seconds while the rocket travels from point B to some point C. Draw in the shape of the path from B to C. (Show your best guess for this problem even if you are unsure of the answer.)

b) Show the path from point C after the engine is turned off on the same drawing.

Student's Drawing:



Physicist's Answer:



Typical Incorrect Answer: The force of the rocket engine combines with whatever was making it go from A to B to produce path BC. After C, whatever made it go fom A to B will take over

> and make it go sideways again causing the rocket to return to its original direction of motion.

Apparently, the presence of the initial constant motion necessarily implies the presence of a propelling force for the student (even though the problem explicitly states otherwise.) The curved parabolic path from B to C is a detailed aspect of the motion that the uninitiated student will rarely reproduce. A more significant difficulty than this, however is the tendency in many students to draw the rocket's motion returning to a horizontal direction after the engine is shut off at point C. These students typically state that "whatever was making it go from A to B will make it go sideways again" after C. The student's prediction of the return of the rocket to a horizontal path indicates that the student believes in some influence acting in the rocket from A to B which "takes over" again after C. This indicates that, for the student, the presence of constant motion from A to B necessarily implies the presence of an external propelling force, even though the problem states that no outside forces are present.

SUMMARY OF CHARACTERISTICS FOR THE "MOTION IMPLIES A FORCE" PRECONCEPTION

By studying the previous examples, we can build a preliminary model of the typical student's preconceptions in this area. The following list summarizes what appear to be the most common characteristics of the "motion implies a force" conceptual system.

(C1) Effects of Force

($C1_A$) In this system, any motion, even at a constant velocity, triggers an assumption of the presence of a force to cause it. This is illustrated in Example 4, the Rocket Problem. In particular, a continuing force is required for continuing motion.

(Cl_B) The direction of an object's motion is instantaneously associated moment by moment, with the direction of force on an object.

(Example 3:Forces on Pendulum.)

(Cl_C) "Impetus" type forces are inferred to explain phenomena which might otherwise contradict (Cl_A) such as motion which continues in the face of an obvious opposing force. (Coin and Pendulum Problems.) This force is often called the "force of momentum" or simply, "the force that makes it go up."

(Cl_D) In some cases the quantity of motion is assumed to vary instantaneously with the quantity of force. (Example 3:Cart's Maximum Speed). (Other students recognize the presence of a delay in the build-up of speed from rest in this problem but still seem to believe (Cl_A), (Cl_B), and (Cl_C).

In summary, a certain pattern emerges when one studies the responses of the naive student to elementary problems in dynamics. This pattern suggests the presence of a system of preconceptions summarized by the phrase "motion implies force". The wide diversity of situations shown here in which this system of preconceptions surfaces is indicative of its pervasive nature. This suggests that the system is deep seated and is one source of the difficulties encountered by students in understanding the physical principles associated with the equation $F = ma$.

GROUP TESTING

In order to investigate the extent to which these misconceptions are widespread we gave written versions of the Rocket and Coin problems to two groups of engineering students at a large state university. The data are shown in Fig. 1.

	Rocket Problem		Coin Problem
	Part A	Part B	
Incorrect Answers Pre-Physics	134 89% (n=150)	93 62%	30 88% (n=34)
Incorrect Answers Post-Physics	17 71% (n=24)	5 21%	18 75% (n=24)

Fig. 1

Response Categories for Rocket Problem n = 150

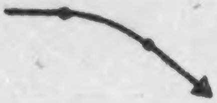
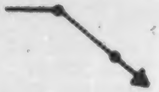
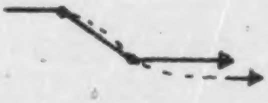
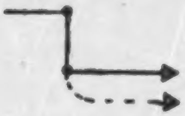
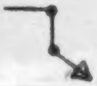
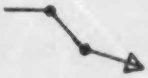
		Entering Freshman Engineers	
1) (Correct)		14	9%
2) (Partially Correct)		40	27%
3) (Returns to Horizontal)	 	62	41%
4) (Returns Partially to Horizontal)	 	8	5%
5) Other		26	17%

Figure 2

Pre-physics group. The "naive" group of students were given the problems on a diagnostic test early in their first semester in a class required of all engineering majors. These students had not had college physics, but most had had high school physics. In general the beginning students did very poorly on these questions. 89% drew an incorrect path for part A of the Rocket problem while 62% missed part B. A summary of the responses to the Rocket problem is given in Fig. 2. In addition, interviews were conducted with 18 students solving the Rocket problem. 5 of the 7 students who had responses of type 3 or 4 in Figure 2 mentioned that "what ever was making it go to the right before will take over again after point C." (See appendix for examples of transcripts.)

On the coin problem, 88% of the beginning students gave an incorrect answer. Virtually all (90%) of the errors in this case involved showing an extra force on the coin pointing upwards at position B. See appendix II for a sample transcript from the coin interviews. These findings support our hypothesis that the "motion implies a force" preconception was involved in the beginning students' responses to these problems.

Post-physics group. We also gave these two written transcripts to a group of sophomores who had just completed a course in mechanics. Scores of the post course students were somewhat better but an alarmingly high number of students still gave wrong answers of the same kind on these very basic problems. This was in spite of the fact that the Rocket and Coin problems are qualitative ones requiring none of the quantitative formulae that typify "sophisticated" physics. The post-physics group were paid volunteers who agreed to take a diagnostic test before their final exam in the one semester introductory mechanics course for engineers.

This groups' average grade in the course happened to be significantly higher than the course mean. On the Rocket Problem, these students did somewhat better in avoiding the most blatant error: the misconception that the rocket will return to a horizontal path. However, on the Coin problem, the percentage of error only dropped from 88% to 75%, a rather disturbing result. In the coin problem, all errors were again in the form of an upward arrow. This group was also asked to label the forces drawn. 80% of those who did this still indicated an upward "force of the toss" on the coin. Additional data for this group shows 50% of these students making the same type of error on the Pendulum problem.

It should be noted that a direct comparison between groups cannot be made since the pre-course and post-course tests were given to different groups. However, the two independent results give us some insight into what can be expected of students before and after the introductory course, and the fact that the post-group was an above average group leads us to be concerned about the level of understanding that is generally attained.

In conclusion, the data also support the hypothesis that for the majority of students, the "motion implies force" preconception was not significantly affected by the introductory course in mechanics. This conclusion applies to the extent that they could not solve a basic problem of this kind where the direction of motion does not coincide with the direction of force.

Discussion

We believe that we have identified a major system of preconceptions that many students bring with them into physics courses. We believe that the students using this system of preconceptions hold a different view, not only of the relationship between force and motion, but also the elemental concepts of force, momentum, velocity, and acceleration themselves. We have collected data on students who have completed a course in college physics which indicates that this set of preconceptions can be highly resistant to change -- they do not simply disappear after students are exposed to the alternative view in their physics courses. More likely, these Newtonian ideas are simply assimilated or distorted to fit old conceptions; or they may be blindly memorized as formulas with no connection to deeper qualitative concepts. All of us have probably had the experience of tutoring a student in a help session and having the student nod along, indicating an understanding of our explanation, only to reveal in answering a simple question afterwards, that he has completely misunderstood the point. We wonder whether such communication gaps sometimes occur because the student is attempting to assimilate our remarks in terms of his own preconceptions. Discouraging as this data may seem, one should remember that historically, pre- Newtonian concepts of mechanics had a strong intuitive appeal and scientists were at least as resistant to change as are our students.

Preconceptions need not be viewed exclusively as obstacles to learning. Since they ordinarily have some predictive power in certain practical situations, they can be thought of as "zeroth — order models" which need to be modified in order to achieve greater precision and generality. See Clement (1979) for a discussion of how certain intuitive conceptions constitute a foundation on which more precise, quantitative principles can be built.

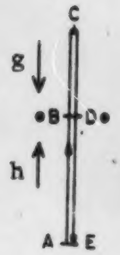
The stability of these preconceptions suggests that intellectual growth in this area of physics is only likely when the student actively uses his preconceptions and can see precisely where they lead to irreconcilable contradictions. This suggests that we find teaching strategies that encourage students to make qualitative predictions based on their own preconceptions, and to make explicit comparisons between these intuitions, Newtonian explanations, and convincing empirical observations. See Clement (1977) for one example of such an attempt.

It is therefore important that we each try as physics teachers to become more sensitive to the preconceptions at work in our students. This is perhaps more easily done at the beginning of the tutoring sessions, where one can draw out intuitive views by encouraging students to think out loud about qualitative problems. This requires a certain amount of patience, openness and respect for the student's views. In order to elicit the student's own views one must be an interested listener part of the time and an active teacher at other times. During these "listening" episodes, one must postpone the goal of teaching temporarily and resist the temptation to correct the student at every opportunity.

Once these conditions are met, most teachers are surprised at the number of common preconceptions that can be observed. Some of these are specific to particular problem situations; but many are quasi-consistent and stable with respect to a variety of problems. Almost all students will have intuitive opinions and predictions to make when presented with elementary qualitative problems.

We cannot consider the student to be a blank slate in these areas. It is important to recognize that the concepts we present must displace intuitive concepts that the student has constructed over a great many years. Increased awareness of these preconceptions should allow us to develop new instructional strategies which take student views into account and which foster a much deeper level of understanding than is currently the norm.

EXAMPLES OF TRANSCRIPTS



Coin Toss Problem

S: "A coin tossed from point A straight up into the air and caught at point E. On the dot to the left of the drawing, draw one or more arrows showing the direction of each force acting on the coin when it is at point B. Draw longer arrows for the longer forces."

S: So the force going up and there is the force of gravity pushing it down and the gravity is less because the coin is still going up until it gets to C.

I: Okay, you want to label those for me?

S: Un, just write gravity next to the top one there.

I: What kind of name should we give the other one?

S: Force of the throw.

I: Okay, now I would like you to say a little more about each one and also say whether you think one is stronger than the other.

S: I guess there's --- if the dot goes up the force of throw gets to be less and less because gravity is pulling down on it, pulling down.

I: Okay, what about the length of this arrow, if we use that to represent how strong the force is, you think it would be stronger than gravity at point B ?

S: Yeah, because the ball is still going up, so the force of the throw is still overcoming the force of gravity that wants to make it go down.

I: Okay, what about at C?

S: At C, they are just equal and uh, that's as much energy as the force of the throw can lift, the force can lift the dot before the force of gravity makes it go down again.

I: Uh huh, okay. So the two forces would be equal at C.

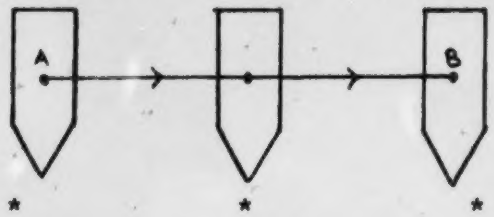
S: Yeah.

Examples of Transcripts from the Rocket Problem

Students were interviewed working on the following problem:

a) A rocket is moving along sideways in deep space, with its engine off, from point A to point B. It's not near any planets or other outside forces. Its engine is fired at point B and left on for two seconds while the rocket travels from point B to some point C. Draw in the shape of the path from B to C.

b) Show the path from point C after the engine is turned off on the same drawing.

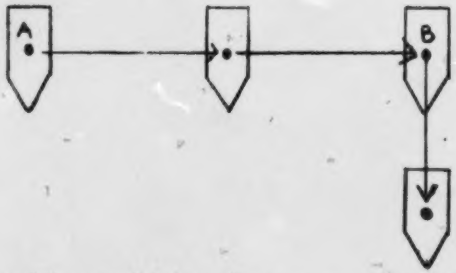


Kerry answered this question in the following way:

Kerry) It would go down. . . .

I) Could you draw it in?

(draws):



Kerry) I would say this would go that way. (points down)

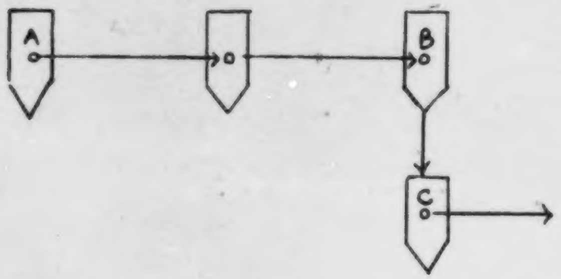
I) And continue that way?

Kerry) Well — when you shut it off, it would start drifting again, (motions horizontally to the right) wouldn't it?

I) OK, so if I burn the engine to this point and then shut it off -- how would you draw what would happen after I shut it off?

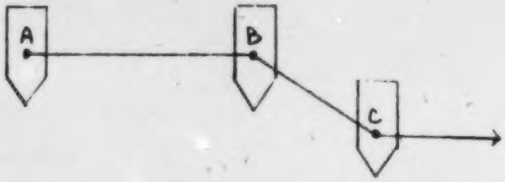
Kerry) OK.

(draws):



Kerry apparently has very different intuitions than the physicist about how the original motion and the motion caused by the rocket engine will add in this case. When the rocket fires, she changes its direction instantaneously, indicating that she thinks in terms of a direct relationship between applied forces and resulting motion. And when the rocket shuts off, its original motion mysteriously reappears in her drawing.

Bob, another student, draws:



I) OK, can you describe the motion and tell me what the rocket did?

Bob) OK. The rocket was moving towards here (from A to B) -- a force acting upon it here (point of ignition, B) to drive it down -- so in effect it would be driving it at an angle because there's two forces acting upon it -- it'd be in an unbalanced situation -- it'd move to where — to where it wouldn't be opposing -- it'd move to an angle.

I) And after the engine shuts off?

Bob) Right here (points to C) -- and with the same force acting upon it -- motion -- it'd continue along this path (horizontally to the right).

Bob apparently associates the initial horizontal movement of the rocket with a force that will continue "acting upon it".

APPENDIX III

TRANSLATING BETWEEN SYMBOL SYSTEMS: ISOLATING A COMMON DIFFICULTY IN SOLVING ALGEBRA WORD PROBLEMS*

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March, 1979

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ABSTRACT

Many science oriented college freshmen cannot solve a particularly important kind of algebra word problem. The major source of difficulty is the translation process between words and equations; it is not in the ability to comprehend English or manipulate algebra. Meaningful translations between symbol systems require a more complex process than previously recognized.

Humans and certain other primates can use a variety of different symbol systems such as spoken language, sign language, mime, writing, and mathematics. It is often tacitly assumed that if an individual has mastered syntactical rules within each of these systems he will be able to translate between any two of them, but there are reasons to believe that this is not true. For example, foreign language instruction emphasizes grammar and vocabulary; yet many grammatically correct translations by translators not familiar with the subject matter do not convey the appropriate meaning. Also in machine translation of natural languages, purely syntactic translation algorithms have proved to be inadequate to the task. (1)

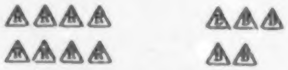
Paige and Simon have shown that many people depend on syntactic strategies when they translate English word problems into algebraic equations, but that while these rules are adequate for some problems they can produce incorrect or meaningless results in others. (2) The data we present confirm these findings and expose a class of problems which should be trivial for a scientifically literate person but which are solved incorrectly by large numbers of science-oriented students.

Table 1 shows selected problems from a 45 minute, written test that was given to 150 freshman engineering students at a major state university. The test was administered during a regularly scheduled class period early in the first semester. Subjects were told that their performance would not affect grades but that the test would help us determine how to improve engineering instruction. All appeared to take the test seriously and all finished their work in the allotted time.

Items 1, 2, and 3 were designed to test algebraic skills. For each problem, over 90% of the students were able to manipulate these algebraic expressions correctly. Items 4, 5 and 6 tested the ability to read written English and translate it into a representation suitable for simple numerical

Table 1

Test Questions (n = 150)

	Correct answer	% correct	Typical wrong answer
1. Solve for x: $5x = 50$	$x = 10$	99	
2. Solve for x: $\frac{6}{4} = \frac{30}{x}$	$x = 20$	95	
3.* Solve for x in terms of a: $9a = 10x$	$x = \frac{9a}{10}$	91	
4.* There are 8 times as many men as women at a particular school. 50 women go to the school. How many men go to the school?	400	94	
5. Jones sometimes goes to visit his friend Lubhoft driving 60 miles and using 3 gallons of gas. When he visits his friend Schwartz, he drives 90 miles and used _?_ gallons of gas. (Assume the same driving conditions in both cases.)	4 1/2	93	
6. At a Red Sox game there are 3 hotdog sellers for every 2 Coke sellers. There are 40 Coke sellers in all. How many hotdog sellers are there at this game?	60	93	
7. Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this University." Use S for the number of students and P for the number of professors.	$S = 6P$	63	$6S = P$
8. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who order cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered.	$5C=4S$	27	$4C = 5S$
9.** Write an equation of the form P_A = ____ for the price you should charge adults to ride your ferry boat in order to take in an average of D dollars on each trip. You have the following information: Your customers average 1 child for every 2 adults; Children's tickets are half-price; Your average load is L people (adults and children). Write your equation for P_A in terms of the variables D and L only.	$\frac{6D}{5L}$	2	[illegible]
10.* Write a sentence in English that gives the same information as the following equation : $A = 7S$. A is the number of assemblers in a factory. S is the number of solderers in a factory.		29	**
11.* Spies fly over the Borun Airplane Manufacturers and return with an aerial photograph of the new planes in the yard.  They are fairly certain that they have photographed a fair sample of one week's production. Write an equation using the letters R and B that describes the relationship between the number of red airplanes and the number of blue planes produced. The equation should allow you to calculate the number of blue planes produced in a month if you know the number of red planes produced in a month.	$5R = 8B$	32	8R

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[*] n = 34 for these problems

[**] n=83 for this problem

[***] Seven solderers for every assembler

calculations; in each case over 90% were successful. Items 7, 8 and 9 tested the ability to perform increasingly complicated translations from English statements into algebraic statements; 98% failed the most difficult problem (#9) while 37% failed the easiest example (#7). The startling drop in performance from 90% to 60% and below suggest that the students' difficulty can be attributed specifically to the translation process.

The errors made on problems 7 and 8 were largely of one kind; in both cases 68% of the errors were reversals: $6S = P$ instead of $S = 6P$ and $4C = 5S$ instead of $5C = 4S$. These reversals might be interpreted as careless errors, except that roughly half of the subjects were given the following hint with both problems: "Be careful: some students put a number in the wrong place in the equation." This hint had no significant effect; it increased the percentage of correct solutions by only 3% and 5%, respectively.

To investigate the source of these reversal errors we conducted audio and video-taped clinical interviews with fifteen students who were asked to think out loud as they worked these and other similar problems. In the "Students and Professors" problem, two basic sources of reversals were identified: a syntactic type and a semantic type. In the first, the student simply assumes that the order or contiguity of key words will map directly into the order of symbols appearing in the equation. For example, one student wrote $6S = P$ and explained, "Well, the problem states it right off: '6 times students'. So it will be six times S is equal to professors."

In the second or semantic type of error, the subject links the equation to the meaning of the problem. However, the equation is seen not as an expression of equivalence but as a description of relative size. To students using this approach, the fact that the "S" side of the equation has a 6 on it indicates that it is larger than the "P" side which has no modifier. Thus, there appear to be more S's than there are P's. For example, one subject wrote

6S = 1P and explained "There's six times as many students, which means it's six students to one professor and this (points to 6S) is six times as many students as there are professors (points to 1P)." When asked to draw a picture to illustrate his equation, the student drew from right to left, one circle with a 'P' in it, an equal sign, and six circles with S's in them. (3) Such subjects interpret the incorrect equation as stating that a large group of students are associated with a small group of professors. In this interpretation the letter 'P' apparently stands for "a professor" rather than "the number of professors" and the equal sign expresses a comparison or association, rather than an equivalence. Thus, although these subjects have an accurate semantic conception of the practical situation, they still fail to symbolize that understanding with the correct equation (see figure 1).

In some protocols, subjects wrote down the correct equation, but then switched to the reversed form. This indicates that for these students the reversed equation is the more compelling one.

In a follow-up study questions 10 and 11 were given to a separate group of 34 students from the same population. About seventy percent of the students produced incorrect answers when translating from an equation to words or from a picture to an equation and over 75% of the errors were reversals. In problem 11 it is difficult to attribute these errors to simply a syntactic strategy; the semantic reversal described above is a more plausible explanation.

It is important to stress that these students have no difficulty in reading English. They are skilled in the manipulation of simple algebraic equations, but when asked to invent a simple equation for a situation they can experience some difficulty. What they cannot do is translate between the two symbolic systems. Most can translate from simple, verbal statements to an equation in one variable, such as (for problem 5):

	Syntactic	Semantic	
Method	*Word Order Match*	*Size Comparison*	*Operational Equality*
Answer	6S = P	6S = P	S = 6P
Result	Incorrect	Incorrect	Correct
	Passive		Active Operation

Fig. 1. Solution Methods for "Students and Professors" Problem

$$\frac{60}{3} = \frac{90}{x}$$

but many have difficulty with very simple expressions in two variables.

The structure of the correct translation process is exposed when we clarify certain tacit assumptions underlying conventions in algebraic notation. The correct equation, $S = 6P$, does not describe sizes of the groups in a literal or direct manner; it describes an equivalence relation that would occur if one were to perform a particular hypothetical operation, namely making the group of professors six times larger than it really is. Some students find the correct equation by writing the reversed equation first and then plugging in numbers as a check. However, analysis of protocols from successful solutions indicates that the key to understanding the correct semantic translation lies in viewing the number six as an operator which transforms the number of professors into the number of students. One subject who correctly wrote $S = 6P$ said, "If you want to even out the number of students to the number of professors, you'd have to have six times as many professors." The equation is thus read as an instruction to act rather than as a static comparison. In this regard we note that because questions 4, 5 and 6 request a numerical result the subject will, at a minimum, carry out an action in the form of an arithmetic operation. This contrasts with questions 7, 8 and 9, where the operations must be carried out implicitly.

In order to investigate the effect of active and static perspectives we examined a question similar to 8 in the context of computer programming. One might expect that writing a computer program is more complicated and hence more difficult than writing an algebraic equation. However, programming, unlike algebra, induces one to take an active procedural perspective. The programmer should: (1) represent all operations explicitly, (2) view the equal sign (=) as an assignment operator, (3) view an equation as a transformation from an input to an output. We felt this perspective might prevent errors of the form described earlier.

Our subjects, in this experiment, were 17 professional engineers, with 10 to 30 years experience. They were taking a one week course on the BASIC programming language. During the first day of the course they were asked to write an equation for the statement: "At the last football game, for every four people who bought sandwiches, there were five who bought hamburgers." Eight of the engineers failed to correctly solve this problem. On the second day of the course, and without any discussion of the answers to the above question they were asked to write a computer program as follows: "At the last company cocktail party, for every 6 people who drank hard liquor, there were 11 people who drank beer. Write a program in BASIC which will output the number of beer drinkers when supplied with the number of hard liquor drinkers." All subjects answered this question correctly using the statement LET B = (11*H)/6 (or some variant) in their program. The form of this statement is equivalent to that of the correct answer to the first question. The success of the engineers in this computational setting supports our earlier hypothesis that the reversal difficulty is associated with viewing the problem from a static perspective.

Fluent translations between symbol systems such as verbal statements, graphs, programs, diagrams and equations are an essential part of scientific thinking. Investigations of the cognitive processes responsible for these translations are still in an embryonic stage. It is well known that many people cannot solve "word problems." We have identified some specific causes of translation errors that locate an important source of this problem. Students who understand the translations discussed in this paper tend to view equations from an active perspective; that is, they see them as describing the result of one or more operations. We believe that the reason so few students reach this level of understanding stems in part from the lack of emphasis schools place on translating between symbol systems as a separate skill and in part from the static perspective into which much of mathematics is cast. (4) These results provide a disturbing picture of the level of mathematical understanding commonly attained in technical fields, and they suggest that we need to reevaluate some basic assumptions in mathematics instruction.

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Notes

(1) Among the many articles relevant to this question are: Bobrow, D.G. and Winograd, T., "An Overview of K.R.L., a Knowledge Representation Language," *Cognitive Science*, *1*, 1 (Jan. 1977), and Novak, G.S. Jr., "Representations of Knowledge in a Program for Solving Physics Problems," International Joint Conference on Artificial Intelligence, 286 (1977).

(2) Paige, J. and Simon, H., "Cognitive Processes in Solving Algebra Word Problems," in *Problem Solving Research, Method and Theory*, B. Kleinmuntz, Ed. (John Wiley & Sons, N.Y., 1966).

(3) Another student, working from the statement: "There are 8 times as many people in China as there are in England," wrote 8C=1E and said, "It means that there is a larger number of Chinese (points to '8C') for every Englishman (points to '1E')."

(4) Detailed analyses of this problem are given by J. Kaput in "Mathematics and Learning" and A. diSessa in "Learnable Representations of Knowledge." Both papers appear in *Cognitive Process Instruction*, J. Lochhead and J. Clement, Ed. (Franklin Institute Press, Philadelphia, 1979).

(5) We thank R. Narode and N. Fredette for their help in collecting data and F.W. Byron, Jr., S. Polatsek, A. Well and C. Clifton for comments on earlier drafts of this paper.

(6) This work was supported by grants from the National Science Foundation, NSF-RISE #SED77-19226; the NSF-NIE joint research program, #SED78-22043; and the Army Research Institute for the Behavioral and Social Sciences, ARI #DAH3 19-77-C-1002

APPENDIX IV

SOLVING ALGEBRA WORD PROBLEMS: ANALYSIS OF A CLINICAL INTERVIEW*

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Solving Algebra Word Problems: Analysis of a Clinical Interview

John Clement

Abstract

The ability of science-oriented college freshmen to solve algebra word problems is markedly deficient in certain key areas. An investigation utilizing both written tests and clinical interviews indicates that a major source of these deficiencies lies not in the ability to comprehend written problems, but rather in the ability to translate from words to equations. These findings suggest that the process involved in translating from one symbol system (such as written English) to another (such as algebraic notation) is more complex than has previously been recognized, and that as a skill it has received too little emphasis in educational programs. Further study of student misconceptions promises to make possible the development of psychological models of such translation processes.

This paper presents data which show that a large proportion of science-oriented college students are unable to solve a very simple kind of algebra word problem. In a second section a sample protocol from a student thinking out loud about one of these problems is analyzed in order to identify the specific processes that are responsible for his errors. Our findings have led us to view the nature of the processes underlying the correct use of algebraic symbolization in a new way.

Paige and Simon[1] have shown that syntactic methods, that is, methods not dependent on comprehending the meaning of the described problem situation, are adequate for solving some algebra word problems in one variable. They point out, however, that these methods can produce incorrect or meaningless results in other problems. The results presented here indicate that this finding extends to the case of writing equations in two variables. The data also expose a class of problems which should be trivial for a scientifically literate person but which are solved incorrectly by large numbers of science-oriented students. The analysis of a particularly revealing protocol of a student thinking aloud while solving these problems indicates the structure of the mental processes which cause his solution error.

Test Data

Table 1 shows four problems selected from a 45 minute, written test that was given to 150 freshman engineering students at a major American state university. The test was administered during a regularly scheduled class period in the first semester. Subjects were told that their performance would not affect their grades but that the test would help us determine how to improve engineering instruction. All appeared to take the test seriously and all finished their work in the allotted time.

In examining the results, we see that problem 1 is similar in form to problem 3, except that problem 1 asks for a particular numerical result, while problem 3 asks for a general equation. The same is true for problems 2 and 4. The contrast between the number of students who correctly solve the numerical vs. the algebraic problems indicates that the students have a specific difficulty in translating from words to equations.[2]

Table 1

Test Questions (n = 150)

	Correct answer	% correct	Typical wrong answer
1.* There are 8 times as many men as women at a particular school. 50 women go to the school. How many men go to the school?	400	94	
2. At a Red Sox game there are 3 hotdog sellers for every 2 Coke sellers. There are 40 Coke sellers in all. How many hotdog sellers are there at this game?	60	93	
3. Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this University." Use S for the number of students and P for the number of professors.	$S = 6P$	63	$6S = P$
4. Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there are five people who ordered strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered.	$5C = 4S$	27	$4C = 5S$

* n = 34 for this problem, n = 150 for others

The errors made in problems 3 and 4 were largely of one kind; in both cases 68% of the errors were reversals: 6S = P (or an algebraically equivalent statement) instead of S = 6P and 4C = 5S instead of 5C = 4S. These reversals might be interpreted as careless errors, except that roughly half of the subjects were given the following hint with both problems: "Be careful: some students put a number in the wrong place in the equation." This hint did not have a significant effect on the percentage of correct answers. The performance of the group given the hint was only slightly better: the percentage correct was 3 points higher on problem 3 and 5 points higher on problem 4.

Protocol Data

While these written tests are useful in establishing the existence of a serious difficulty in this general area, the data they provide is too coarse to give us insights into different mental processes at work in the students which lead to correct and incorrect answers. In order to develop hypotheses concerning the internal mechanisms behind these results we conducted audio and video-taped clinical interviews with fifteen students who were asked to think out loud as they worked on these and other similar problems.

These revealed two basic sources of reversal errors, one syntactic and the other semantic. In the first or syntactic type the student simply assumes that the order of key words in the problem statement will map directly into the order of symbols appearing in the equation. In the second or semantic type of error the subject appears to comprehend the meaning of the problem situation. However, the equation is seen not as an expression of equivalence but as a description of relative sizes. To these students, the "S" side of the equation 6S = P has a 6 indicating that it is larger than the "P" side which has no modifier. Thus there appear to be more S's than there are P's. The student feels that the equation then symbolizes the intended situation of having a large group of students and a small group of professors. Thus the student may have an accurate picture of the relative sizes of groups in the practical situation, but still fail to translate his or her understanding correctly into an equation. Analysis of protocols from successful students indicates that the key to performing correct translations lies in the ability to conceive of a mental action that produces an equivalence, and to realize that it is precisely this action that is symbolized in the equation. (See fig. 1) These various approaches to the problem will be described more fully below.

In this paper we will examine a particularly revealing protocol of a single student who vascillates between a correct and an incorrect approach to the problem. This allows us to illuminate and compare each of the processes outlined above. The student, referred to here as Tom, was first asked to solve the following problem:

> China Problem
> Write an equation for the following statement, using the variables A, C, and R: "There are one third as many people in America as there are in China and Russia combined."

His final answer to this problem was 1/3A = C + R. (A correct answer is A = 1/3 (C + R)). Thus he ends up with the multiplicative factor of one-third on the wrong side of the equation.

He also worked on the less complicated "Students and Professors" problem and made a very similar error, as shown below.

> Students and Professors Problem
> Write an equation for the following statement: "There are six times as many students as professors at this university." Use S for students and P for professors.

Here Tom wrote: (Eq.1) $6S = 1P$

(Eq.2) $S/6 = P$

(Eq.3) $S = 6P$

(However, he then rejects Eq. 3 and points to Eq. 1 as his final answer)
(The correct answer is $S = 6P$)

We can say in this case that Tom fails to translate from a verbal description of a practical relationship to an equation. These errors are more striking when we consider the fact that he was doing B+ work in a standard calculus course at the time of the interview. He was able during the same interview to find the derivative of the function $f(x)=\sqrt{x^2 + 1}$, rapidly differentiating it by using the chain rule.

In both problems Tom gives a "reversed" equation for his answer. What is the nature of the conceptual difficulty that is responsible for this error pattern? One way to account for the errors is to simply assume that Tom was careless--that he just did not try hard enough or became confused and made a random mistake. However, the fact that he misses more than one problem in this same way indicates that he makes this mistake consistently. In addition, he actually rejects a partially correct answer in his solution attempt above. Therefore we cannot explain his error by simply saying that it didn't occur to him that the equation could be written in more than one way. He seems to have a real difficulty in determining the validity of his translation from the verbal description to the equation.

It is interesting to note that most students do not miss the same problem when it is phrased in the following way: "The number of students is equal to six times the number of professors." Most students correctly write $S = 6P$. Since the temporal order of saying the verbal statement and writing the mathematical equation are the same in this case, this suggests a possible hypothesis: that erring students perform a direct syntactic translation rather than a meaningful translation. Presumably such a syntactic translation cues off of the word "times" in the original problem to indicate a multiplication and simply preserves the order in which groups ("students" and "professors") and numbers occur in the sentence--no matter which way the sentence is constructed. This hypothesis in fact can account for the behavior in a significant number of the reversed solutions observed in clinical interviews. However, other protocols do not fit this hypothesis, such as the transcript which shows Tom's behavior in solving the Students and Professors problem. It is suggested that the reader read through the transcript at this point. (Appendix I)

Protocol Analysis

As shown in the transcript, Tom actually gives a correct solution in line 20--only to reject it in favor of his original incorrect solution. We will analyze each section of the protocol in terms of the characteristics of cognitive processes at work in Tom during that section in order to attempt to explain his anomolous behavior.

In section A of the transcript Tom writes down the incorrect equation $6S = P$ and the initial analysis problem is to explain why he does this. In particular, is the theory correct that he simply makes a syntactic translation from the English statement to the written equation? Certainly it is plausible that such a translation takes place to provide Tom with an initial hypothesis for the equation to be written. This type of translation might be likened to the simple act of paraphrasing a long sentence in short hand form by copying the main elements, in the order in which they appear, and dropping out the inessential words. Such a translation might be performed with little or no understanding of meaning of the sentence. The student simply assumes that the order or contiguity of key words will map directly into the order of symbols appearing in the equation.

However, Tom's reference in line 8 to a "ratio of 6 to 1" and his comments in later sections indicate that a purely syntactic translation is not the only cognitive process occurring in the interview. We can infer from lines 8 and 14 that in addition, he has a semantic conception of a large group of students and a much smaller group of professors. This relationship would presumably be supported by his practical knowledge of a typical university.

The equation 6S = 1P appears then to be an incorrect but meaningful way for Tom to symbolize the relative sizes of the two groups--the appearance of the "6S" on the left side indicating a large group of students and the solitary 1P on the right indicating a much smaller group. This interpretation is consistent with the somewhat unusual way in which he includes the word 'one' in his statement: "Six 'S' equals one 'P'". Thus Tom's reversed equation appears not to be the product of a purely syntactic approach but to be an error produced by a process with a semantic component. Tom is also able to rephrase the original statement of the situation in line 14, which indicates that he has a semantic conception of the described relationship between students and professors that is more than just an ability to "parrot back" the original statement.

In part B of the transcript he is also able to temporarily generate a correct translation from his semantic conception. In lines 20-24 he writes 6S = 1P and explains, "There's six times as many students, which means it's six students to one professor and this (points to 6S) is six times as many students as there are professors (points to 1P)." In a later session, when asked to draw a picture to illustrate his equation, he draws, from right to left, one circle with a 'P' in it, an equal sign, contiguity of key words will map directly into the order of symbols appearing in the equation.

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Finally, Tom's behavior temporarily indicates the nature of a third kind of process involved in understanding the correct translation. In part B, Tom appears to consider an approach to the problem which gives him a correct answer, but he eventually rejects this approach in favor of his original wrong answer. In lines 18-20, Tom writes a correct equation, S/6 = P, saying "If you wanted to get like an equal number between the two, you'd have like, ah, how can we do this--S divided by six." We infer that Tom is mentally performing an action of dividing one of the groups (students) into parts here in order to obtain a one to one correspondence between a group of students and a group of professors. As an internalized action this process contrasts with a simple relative size comparison via a static image. (see Fig. 1)

Tom's two approaches to the problem expose the tacit assumptions and meanings underlying our conventions for algebraic notation: his correct equation, S/6 = P, does not describe the situation at hand in a literal or direct manner; it describes an equivalence relation that would occur if one were to perform a particular hypothetical action, namely, making the group of students six times smaller than it really is.

If the above analysis has been successful the reader will now find more plausible the fact that Tom reverts to his initial, incorrect equation at the end of the interview. We chose Tom's solution as a particularly interesting one for analysis because he is a "transitional" case. Tom faces the dilemma of having two apparently meaningful but opposing ways to write an equation for the given English statement. His first equation, $6S = 1P$, symbolizes the iconic, relative size aspect of the situation in a way that is officially forbidden but probably meaningful to Tom. (He is apparently unaware that this style of symbolization is "illegal"). His second equation, $S/6 = P$, correctly symbolizes the equality that would result from a potential action. But his first approach is "stronger" within him and dominates his second approach. Thus Tom's "transitional" solution illuminates both the nature of his misconception and the nature of the conceptions he uses to understand the accepted format for writing equations.

Summary

Our data from group testing indicate that large numbers of science oriented students have difficulty in translating from words to equations of this kind. The most common error is a reversal error which appears to have two sources: a syntactic, word order matching process and a semantic, relative-size symbolization process.

Tom's behavior suggests that in order to symbolize a practical mathematical situation by writing an equation, one must be able to envision one or more operations on the situation if one is operating on the basis of an understanding. Understanding how to translate from words to equations involves both the process of performing a mental action on the situation to produce the equivalence and a comprehension of the way in which standard algebraic notation symbolizes this action. Our findings indicate that a large number of students have not learned to do this reliably in high school. An analysis of 15 thinking-aloud protocols of students working on this and other similar problems has indicated to us that the two types of reversal errors and the correct operational approach are indeed typical.

The high failure rate among college engineering students with such elementary problems indicates a much more severe difficulty in the area of translation skills than we had previously recognized. We suspect that such errors are part of a larger pattern in which students are much more successful in learning rules for performing syntactic manipulations within particular symbol systems than they are in learning to translate between symbol systems in a meaningful way. We have shown that the analysis of protocols using simply-constructed problems can allow us to "home in on" the specific translation difficulties involved. The identification of these specific stumbling blocks will make it easier to design instructional strategies which will overcome them.

APPENDIX

Calculus Problem

Tom's correct solution to the problem of differentiating the function $f(x) = \sqrt{x^2 + 1}$ appears below.

$$\sqrt{x^2 + 1} = (x^2 + 1)^{1/2}$$

$$= 1/2(x^2 + 1)^{-1/2} D_x(x^2 + 1)$$

$$= 1/2(x^2 + 1)^{-1/2} (2x)$$

$$= x(x^2 + 1)^{-1/2}$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Tom: Students and Professors Problem

Session 1

Part A (Verbatim unedited transcript from video tape)

1 I: The problem is to write an equation for the following statement; and the statement is: "There are six times as many students as professors at this university."

2 S: Six times as many students as professors-- (writes 6)

3 I: So let's use S for students and P for professors.

4 S: S for students (writes $\begin{matrix} S & P \\ \text{Students} & \text{Professors} \end{matrix}$) --is that large enough?

5 I: That's fine, yeah, P for professors.

6 S: P for professors

7 I: P for professors.

8 S: I don't know if I spelled that right or not--you got a ratio of six to one (writes $\begin{matrix} S & & P \\ 6 & \Rightarrow & 1 \end{matrix}$)

9 I: Yeah.

10 S: S to P.

11 I: So the problem would be to write an algebraic statement, an equation, using S and P, that represents that statement.

12 S: Ok, just ah, 6S equals ah, P, ah, 1P (writes $6S=1P$) (Equation 1)

13 I: Ok. Any particular reason you write it that way?

14 S: Well if there's ah, there's six, say there's six to one, six times as many students, which means its six students to one professor.

15 I: Mmhm.

16 S: Ok, and, ah, you can just write it out--six times as many students as professors.

17 I: Ok.

Part B

18 S: Now, let me see here. Ok, if you wanted to get like an equal number between the two you'd have like, ah, how can we do this--S divided by 6--I'm just trying to figure out if, ah, like if you wanted ah, to figure out the---what do we want to do here--

19 I: Ok, write that one down and then we can make a--

20 S: --so S over 6 equals P (writes S/6=P) (Equation 2) which means that there would be six 'P' for 'S' (writes S=6P) (Eq.3). We said there's 6 times as many students as professors, which means if you want to, ah, even out the number of students to professors you'd have to have 6 times as many professors--I guess that's what I was trying to think to myself.

21 I: Even out? (S: Yeah) I: Can you rephrase that--show me with the equation?

22 S: Even out the number of students--a one to one correspondence like--so as many students as you have professors.

Part C

23 I: So here you have S and P--

24 S: Uh-huh. This means there are six times (points to 6P in Eq.3) as many professors as there are students (points to S) and this is six times as many students (points to 6S in Eq.1) as there are professors (points to 1P) I was just gonna even them out, I guess.

25 I: Ok.

26 S: So, you have 'em like a one to one ratio, so you get S equals P.

27 I: So how about--which equation would be true for the original statement?

28 S: This one right here (points to '6S=1P')

29 I: The original statement was that there are six times as many students as professors at the university.

30 S: Right there. 6S equals 1P.

31 I: The first one?

32 S: Yeah.

33 I: Ok, and then what would this (points to 'S=6P') be for?

34 S: Six, ah, six times as many professors as there are students.

35 I: Ok, all right.

Student's written work:

$$\begin{array}{ccc} S & & P \\ 6 & \Longrightarrow & 1 \end{array}$$

S	P
Students	Professors

(Eq. 1) $6S = 1P$

(Eq. 2) $S/6 = P$

(Eq. 3) $S = 6P$ (Student points to equation 1 as his final answer)

Tom: Students and Professors Problem
Session 2

In a later session the interviewer decides to try a simple tutoring strategy to see whether Tom will easily change his method of writing equations.

1 I: Let's see. Uh, let's look at the one we did--didn't we do one about students and professors last time?

2 S: Uh--

3 I: Well, I'll give it to you again. The statement you have to write the equation for is: "There are 6 times as many students as professors at this university". I thought we worked on that one?

4 S: Yeah, I think so. (Writes 6S = P).

5 I: Ok. And I think we went through the other way to write it, uh, "S = 6P" and I think we had to decide--between those. Write down "S = 6P".

6 S: Okay. (writes S = 6P)

7 I: Now, I'd like you to draw--try to dray a picture. That seems to be the hint in these problems in deciding which one is the correct equation. We've run into a lot of students who have had trouble with writing equations from these statements. And it seems like drawing a picture sometimes clears up the problem. So, is there any way you could represent with circles or something--the number of professors and the number of students, just the relative sizes?

8 S: Okay, you said there were 6 times as many students as there was professors.

9 I: Right.

10 S: So, say there was one professor (draws circle with 'P' inside) I'll put a P inside there--(writes '=' next to Ⓟ), and we said there's 6 students for every professor, so we could just correspond that by putting 6 circles, all with S's in them. (Draws 6 circles to left of '=' sign, putting an S in each)

11 I: Okay.

Student's Written work:

6S = P

S = 6P

ⓈⓈⓈⓈⓈⓈ = Ⓟ

Ⓢ = ⓅⓅⓅⓅⓅⓅ

12 S: And that would seem to say the same thing; whereas this one (points to !S=6P') would be just the opposite as--students and professors. (Draws Ⓢ = ⓅⓅⓅⓅⓅⓅ) That would be a pretty good ratio in a, in a school--

13 I: Because there would be so many (S: Professors, yeah) teachers for each student? Yeah, okay, now the question is which one do you think is the correct one?

14 S: This one (points to the 1st equation, '6S=P').

Notes

1. Paige, J. and Simon, H., "Cognitive Processes in Solving Algebra Word Problems," in Problem Solving Research, Method and Theory, B. Kleinmuntz, Ed. (John Wiley & Sons, N.Y., 1966).

2. See J. Clement, J. Lochhead and E. Soloway, "Translating between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems." Working Paper, Cognitive Development Project, Department of Physics and Astronomy, University of Massachusetts, 1979, for a discussion of more problems of this kind.

3. For an analysis of other protocols involving this problem see Clement, John, "Solving Algebra Word Problems: Identifying Misconceptions via Protocol Analysis" (in preparation).

4. I wish to thank R. Narode, N. Fredette and K. Quinton for their help in data collection and J. Lochhead, E. Soloway, A. Well, S. Pollatsek, C. Clifton and Frederick Byron for their helpful discussions.

SOME TYPES OF KNOWLEDGE USED IN UNDERSTANDING PHYSICS*

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April, 1978

There is a growing concern among researchers in the area of physics teaching that there has been an overemphasis on teaching introductory physics as a set of equations and principles, linked together in a formal, deductive system. It has been suggested that this approach may contribute to the problem of students learning physics by memorizing large numbers of formulas with little real understanding of the principles behind them. This paper outlines an alternative to the formal deductive system as a model for the nature of understanding in physics.

At Berkeley, Fuller, Karplus, and Lawson argue in a recent article in Physics Today that many college students do not possess the formal reasoning skills required to learn physics directly in terms of a deductive system expressed in symbolic equations. Instead, students learn to manipulate formulas in a superficial manner using the less sophisticated forms of reasoning available to them. They argue for the development of introductory courses for this population of students, that "focus on the development of reasoning rather than the mastery of content."[1] Larkin, also at Berkeley, compared the problem solving behavior of novices and experts.[2] She finds different problem solving styles even when both novice and expert are familiar with the same set of equations. She proposes the hypothesis that experts have complex knowledge structures, in addition to principles in the form of equations, that allow them to apply relevant equations to a problem in a more organized fashion.

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In a report from the LOGO group at M.I.T., diSessa writes:

> In the past, axiomatics or other formal systems have, principally by default, served as model representations of knowledge for pedagogical purposes. But while such systems which stress internal simplicity and coherence may serve useful roles for some purposes, they are not good models for learning. ... We must better take into account intuitive and other formally ill-formed knowledge that students already possess.[3]

Thus diSessa also argues against using a formal and deductive approach to teaching introductory physics.

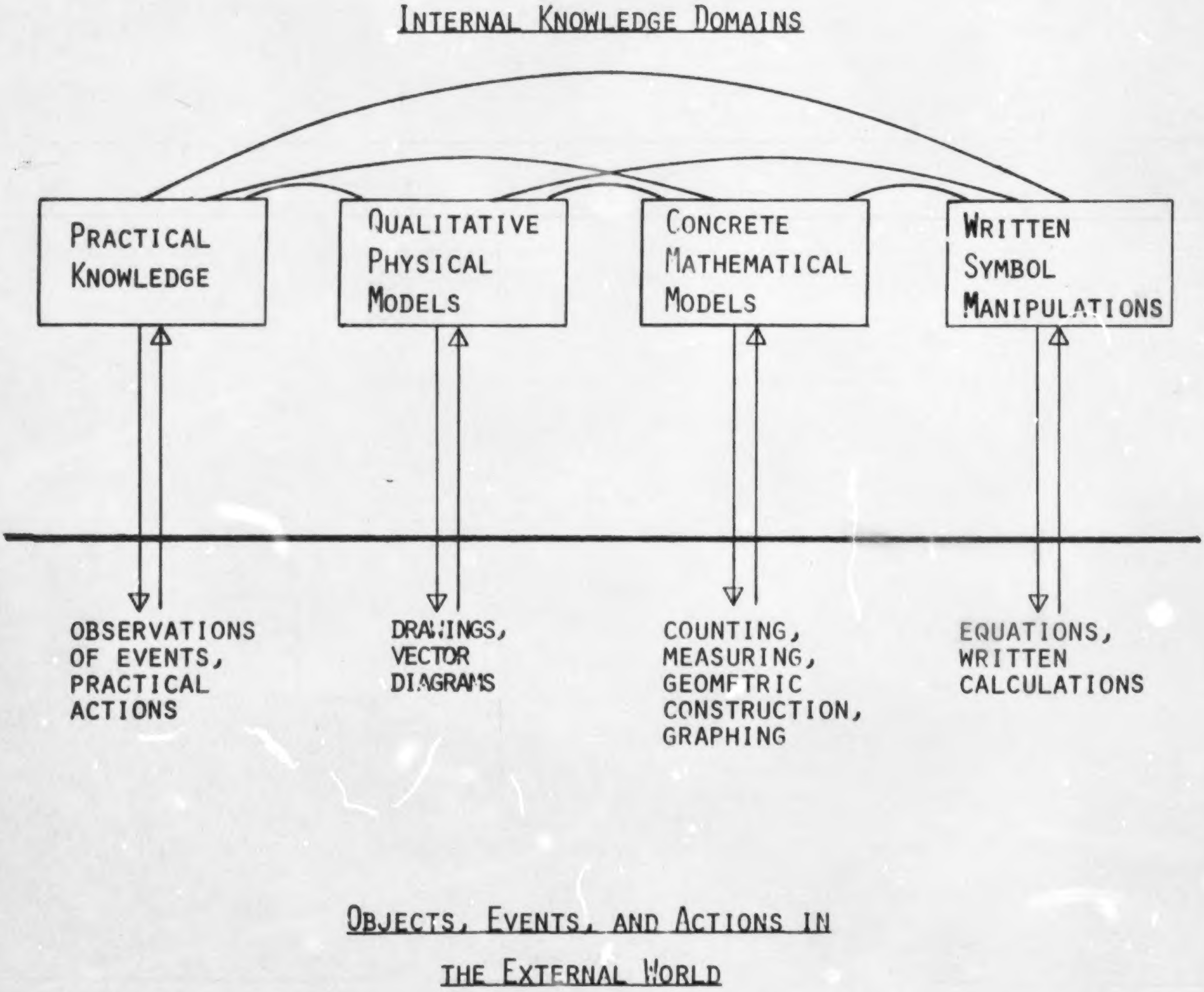
But if physics principles summarized in the form of symbolic equations do not by themselves constitute an effective understanding of physics, then the question arises: "What constitutes a more valid cognitive model of what it means to understand a topic in physics?" If the equations are not the only thing one needs to know, then what are the other key ingredients for understanding?

A Model for Understanding

The diagram on the following page is an attempt to model the several types of knowledge needed for a person to understand a topic in physics.[4] The four large areas above the horizontal line represent four domains of internal knowledge structures in the person, while items below the line represent objects and events in the external world. Thus I want to model types of action-oriented knowledge structures as they actually exist and operate in a person; I do not want to assume, *a priori*, that these are equivalent to the body of knowledge statements--in the form of expressions that can be written down on paper--that comprise a formal exposition of the discipline of physics.

The characterisitics of each domain can be introduced by referring

SOME TYPES OF KNOWLEDGE USED IN PHYSICS


INTERNAL KNOWLEDGE DOMAINS
PRACTICAL KNOWLEDGE
QUALITATIVE PHYSICAL MODELS
CONCRETE MATHEMATICAL MODELS
WRITTEN SYMBOL MANIPULATIONS
OBSERVATIONS OF EVENTS, PRACTICAL ACTIONS
DRAWINGS, VECTOR DIAGRAMS
COUNTING, MEASURING, GEOMETRIC CONSTRUCTION, GRAPHING
EQUATIONS, WRITTEN CALCULATIONS
OBJECTS, EVENTS, AND ACTIONS IN THE EXTERNAL WORLD

to the situation described in the following problem:

The electrical energy used by a battery powered water heater varies according to the formula:

$$E = \frac{\Delta t\ V^2}{R}$$

where E = energy used
Δt = time period of use
V = voltage supplied
R = resistance of the heater (assumed to be a constant for a given heating coil)

The heating coil is changed, so that R is cut to 1/3 of its original value. V and Δt are kept the same. What will be the size of the resulting effect on E? Or is this impossible to predict without knowing the specific values of Δt, V, and R? Give a short reason for your answer.

If a person were dealing with a real water heater, and had some knowledge about how to recognize one, turn it on, change the filament, etc., these would be examples of practical knowledge. These manipulations could also be performed mentally in a thought experiment with an imagined water heater in the absence of a real one.

An example of a knowledge structure in the qualitative physical models domain would be a conception of electrons being "pushed" through the heating element and causing the element to heat up by crashing into its molecules. One could represent and manipulate this model in the real world by using drawings or diagrams. These models are often action-oriented and causal -- in them are embedded anticipations like: "If the electrons are pushed harder, they'll come into the element faster, they'll hit the molecules harder, and more heat will be produced."[5]

At this point one can already begin to model what one means by one level of understanding. If "pushing the electrons harder" is connected mentally with a practical knowledge structure for "how to turn the voltage up" on the real water heater, then one has a qualitative model for understanding that practical aspect of the heater. Notice that one might have this understanding without using any quantitative conceptions.

One crosses into the concrete mathematical models domain when using a conception like "the energy released is probably proportional to the push on the electrons; if I double the push, I'll double the energy released." This kind of mathematical model relating scaled variables via the concept of proportionality can be represented in terms of operations on sets of objects or operations on measured line segments, or in a graph. There are many species of idealized, concrete, mathematical objects used in mathematical models, such as the length of a line segment representing the magnitude of a certain physical variable, or the cutting of an object of a certain size into a certain number of equal parts representing a division relationship between two variables. These conceptions can become activated to represent quantitative aspects of the way the water heater behaves.

Finally, a knowledge of memorized equations and rules of algebra and arithmetic resides in the symbol manipulations domain. An equation can itself be treated as an object capable of being transformed via the rules of algebra and related to other equations by knowledge structures in this domain. For example, the equations

$$\text{Energy} = \frac{\Delta t\ V^2}{R} \quad \text{and} \quad \text{Power} = \frac{\text{Energy}}{\Delta t}$$

could be combined algebraically to yield an expression for the power used by the heater. Given the two formulas this could be (and in courses apparently often is) done using symbol manipulation rules without making any connection to the other types of knowledge mentioned--without an appreciation for any underlying meaning.

As another example, consider the solution one sophomore, Student A, gave for the water heater problem. He wrote:

" $E = \frac{tv^2}{R}$

if $R \rightarrow 1/3\ R$, then $E \rightarrow \frac{tv^2}{1/3R} = \frac{3tv^2}{R}$

E becomes bigger times three."

One can account for this student's behavior by assuming that the only knowledge structures participating in the solution are symbol manipulation structures. In contrast, another sophomore, Student B, said: "The energy would probably be more because if you're cutting down the resistance by 1/3, the energy is going to be able to flow in more freely -- it'll go in faster -- so you should get 3 times the energy." These solutions are interesting because they indicate the use of two entirely different types of knowledge to solve the same problem.

Student A uses a knowledge structure in the symbol manipulation domain. He knows that the equality can be conserved when a variable is changed in an equation by changing the other side of the equation in the same way. This method does not depend on the qualitative situation portrayed in the problem.

Student B uses his knowledge of a qualitative physical model for the situation. He imagines a reduction in the resistance causing an increase in energy flow: "... the energy is going to be able to flow in more freely -- it'll go in faster ..." The second student's method does depend on the qualitative situation portrayed.

The symbol manipulation method is useful (and highly efficient) if the student is working from a given formula or set of formulas. However, the physical model is essential when one is attempting to construct a formula or to select an appropriate formula for a new situation. This suggests that someone who can bring both kinds of knowledge to bear on problems understands the subject more deeply than someone who uses either method alone.

Student B first predicts that more energy will be used, then predicts that three times more energy will flow into the bulb. One can account for this behavior by assuming that he also uses a conception of an inverse proportion in his mathematical models domain. Thus he is able to link together structures from at least two domains in bringing them to bear on the problem.

A major aspect of the theory being proposed here is that the ability to link together structures from these different domains is crucial to the understanding of a topic in physics. Some of these links are simply associations learned by rote -- such as the association of a quantity in the mathematical models domain with a particular letter used to symbolize it in the symbol manipulation domain. Other, more significant links are formed when a structure in one domain assimilates a structure in another domain and provides an interpretation for it. An example of such a link was given earlier, where a qualitative physical model involving a conception of "pushing the electrons harder" assimilated a practical knowledge structure for "how to turn the voltage up on the heater." These links are what cause a model at one level to "make sense" as an interpretation of knowledge at another level.

In terms of this model, then, "understanding energy use in the water heater" consists of a knowledge of symbol manipulations that can be performed on the formula, $E = \frac{\Delta t\ V^2}{R}$, connected to knowledge structures in the other three domains -- concrete mathematical models, a qualitative physical model, and practical operations one could perform on a real water heater.

Pedagogical Implications

This theory of understanding involving interacting knowledge domains is supported by a large number of observations made by the author while tutoring physics students. The theory is undoubtedly oversimplified, and many detailed analyses of clinical interviews need to be conducted in order to refine the theory and establish its validity. But it does provide a framework for discussing several interesting pedagogical problems:

(1) To return to the issue of whether the formal exposition of physics content is a sufficient model for what the physics student needs to learn, one can see that formal expositions emphasize heavily the use of written formulas in the symbol manipulations domain. The danger here is that a student may get "stuck" in the symbol manipulation mode -- he may learn a certain set of equations, but not understand their meaningful interpretation in the form of physical models, mathematical models, or practical actions. Making sure that these connections are made is a worthwhile goal and a real pedagogical challenge.

(2) At the University of Washington, George Monk and others have been developing the student's ability to translate freely between modes of describing physical events: from an equation to a graph to a picture to descriptions of a situation in English and back again.[6] This appears to be a promising approach to increasing the student's ability to make connections between knowledge domains.

(3) Knowing a formula is not the same as knowing when to use it. How does one determine when a formula is applicable to a certain practical situation? The ability to do this is crucial for being able to apply one's knowledge of physics to problems in the real world. It is suggested here that qualitative physical models can play a critical role in providing the connection between practical situations and appropriate equations.

(4) One way to increase the emphasis on understanding in a course is to develop the student's ability to answer 'why' questions like: "Why does the energy used depend on the voltage applied to the water heater?" Satisfying answers to these questions often involve qualitative physical models.

(5) Knowledge structures in the qualitative physical models domain can be formal (developed in the school setting) or intuitive. Intuitive conceptions students enter courses with can be deeply seated and difficult to change. Unless a course puts emphasis on dealing with the physical models domain and takes into account the student's intuitions there, the student may have great difficulty in attaching physical meaning to the equations he is learning. This presents another challenging direction for course improvement.

Notes

1. Robert G. Fuller, Robert Karplus, and Anton E. Lawson, "Can Physics Develop Reasoning?" Physics Today, February, 1977, p. 23 - 28.
2. Jill Larkin, "Processing Information for Effective Problem Solving," SESAME Technical Report, U. of California at Berkeley, 1977.
3. Andrew diSessa "On 'Learnable' Representations of Knowledge: A Meaning for the Computational Metaphor." Technical Report: LOGO Project, M.I.T.
4. The four knowledge domains discussed relate to the levels of abstraction discussed in:

   John A. Easley, Jr., "Symbol Manipulation Reexamined: An Approach to Bridging a Chasm." (Address presented at the 3rd annual meeting of the Piaget Society, Philadelphia, PA., 1973.) (Mimeographed)
5. Similar kinds of direction-of-change relations between ordinal concepts like acceleration, velocity and momentum would also be included in this domain. These are distinguished from practical knowledge by the fact that they focus on particular features of a system selected for purposes of analysis.
6. George Monk, "The Assessment and Evaluation of Student Conceptual Development Induced by Two College Science Courses," NSF Proposal submitted by the Mathematics Department, University of Washington at Seattle, 1975.

APPENDIX VI

John Clement

DRAFT
May, 1979

OUTLINE OF POTENTIALLY OBSERVABLE INDICATORS OF UNDERSTANDING IN STUDENTS

Teachers often claim that one of their goals for students is the achievement of understanding, as opposed to concepts learned by rote. Just what is meant by understanding, however, is often ambiguous, as are the observable criteria that might serve to indicate its presence. In this paper, I would like to discuss some observable criteria that can be used to infer the presence of understanding and then go on to present some theories of the unobservable mental processes which allow someone to understand a certain topic.

Potentially Observable External Indicators of Understanding

1. Problem Solving and Predictions

   One can make predictions and solve standard problems in the area. These are seen as more difficult tasks than simply regurgitating facts.

2. Generalizability

   One has some general knowledge, so that one can apply learned principles to some new problem solving situations that are significantly different from those discussed in the course.

3. Explaining in Simple Terms

   One can explain the meaning of concepts by using language that one knew before entering the course -- using one's own terms. This kind of explanation contrasts with those that are merely a verbal rehash of learned jargon.

4. Giving Examples

   One can generate new concrete examples of principles as opposed to simply restating principles.

5. Long-Term Retention

One can remember knowledge over the long term, not simply for purposes of passing a final exam.

Some other indicators of understanding are:

B1. Satisfaction and Conviction

The student has the experience that the concepts studied in the course "make sense" -- they "fit together" for him or her in ways that are satisfying and convincing.

B2. Connections Between Principles

One can describe connections and relationships between principles in the area as opposed to simply listing principles separately.

B3. Translation Between Different Representations

In at least some subject areas, such as mathematics, the ability to translate between different modes of representation such as equations, graphs, data tables, and practical problem descriptions, seems to be an indicator of understanding.

B4. Generating New Hypotheses

An indicator of understanding at a sophisticated level is the ability to formulate new hypotheses or principles.

SEVEN LABORATORIES ON:

(1) QUALITATIVE PHYSICS; (2) THE CONCEPT OF FUNCTION*

Developed for - Physics 190
Spring, 1978

John Clement

Cognitive Development Project
Department of Physics and Astronomy
University of Massachusetts
Amherst, Massachusetts 01003

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* Development of these laboratories was supported in part by National Science Foundation grant #SER 76-14872 and by HEW FIPSE grant #OE G00-76-03206.

Seven Laboratories on:

(1) Qualitative Physics
(2) The Concept of Function

John Clement

These laboratories were developed for a four credit, one semester course designed to prepare students for future courses in the sciences, particularly physics courses. The laboratory consisted of a two-hour session which met once a week. The students enrolled in the course were freshman engineering students.

Rather than emphasizing proficiency with formulas, the laboratories attempt to promote a deeper understanding of fundamental concepts. They also emphasize the process of prediction and testing. The first four labs stress qualitative physics. Our basic aim in these labs has been to have students grapple with ideas of force, displacement, velocity, acceleration, mass and momentum. These concepts are logically tied together in the system of Newtonian mechanics for the physicist, but for the beginning student, they are unfamiliar and in some cases the relationships between them are counterintuitive. We have tried to design a laboratory experience where students can discuss and explore the use of these concepts with concrete objects. The last three labs stress the concept of linear functions. These labs emphasize the role of functions as powerful tools for predicting an infinite number of events. All labs stress the ability to translate between different external representations : numerical data, verbal descriptions of observed events, equations, vector diagrams, and verbal descriptions of causal theories.

The Instructor's Role

In the laboratory the instructor played the following roles:

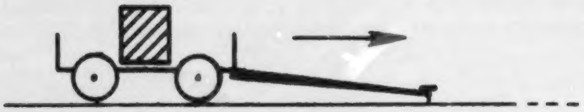
1) helped students find equipment they needed;
2) asked individual groups to verbally describe what they were doing; (this often had the effect of causing students' ideas to become more precise).
3) listened to discussions and observed experiments; (a major part of the instructor's work consisted of trying to understand typical non-Newtonian views of students).
4) encouraged students who were asking questions of their own, designing experiments to test conjectures, or discussing differences in point of view;
5) proposed related thought experiments (ie., "what would happen if there were no friction?" ; "Is this like an automobile accelerating?")
6) verbally labeled heuristic strategies students used ("When you think about sliding the coin on a vertical track that's called thinking about an extreme case"; "when you changed weights on the pendulum keeping the length and initial displacement constant, that's called controlling variables")
7) Communicated conjectures between groups - "Some people argue that the maximum speed of the ball is reached just as you release it on the track, because that's where it's the highest above the table -- others say it's on the horizontal run-off section because the speed takes a while to "build up" -- others say it's somewhere else -- what do you think?")
8) encouraged students to make predictions;
9) Provided a standard label for a concept that a student used when it was perceived that the student's concept was probably a Newtonian one ("When you say 'A sharp increase in speed', the physicists says, 'The acceleration is large there'.")
10) answered students' questions about the Newtonian definitions for the various concepts;

The instructor found it necessary to make a conscious effort to not spend time giving extended explanations of the "correct" point of view. Rather he found it most effective to circulate and spend less than two minutes at a time with each pair of students, returning to each pair several times during the lab period. The instructor provided information to students when they asked for it, but the role with the greatest payoff was seen to be that of stimulating and extending student-student and student-apparatus interactions. Thus for example, the instructor would often ask a pair what they had found, suggest a related question, and then move on to another group without waiting to work through to the answer with them.

Qualitative Physics

Laboratory #1

Horizontal Motion of Cart



Description: A cart is launched by elastic band on a horizontal surface and rolls to a stop.

Dependent Variable: Distance travelled

Possible Independent Variables:
- Strength of elastic
- Mass of cart
- Roughness of surface
- Distance band is stretched

Materials:
- Cart
- Weight Set
- Assorted elastic bands
- 1 meter stick
- 1 wood strip 3/4" x1/8" x .8"
- Thumb tack
- Tape
- Table clamp

1. In column one below, list 3 or more factors you can change in the system, and in column 2 predict whether or not you think each will make a difference in what happens. Where you think changing a factor will make a difference, say carefully what it will affect, and whether it will increase or decrease the magnitude of that variable. You will be graded on how clearly you state your predictions, not on whether your predictions are correct. (In other words, feel free to make intuitive guesses!) Test each prediction and state what happened in column three.

(1) Factor Cha[illegible]	(2) Predicted Effect(s)	(3) Observed Effect(s)

2. Using an entire page draw a schematic of the apparatus used in the experiment. Label the point(s) where the moving object is at maximum speed and where it is at minimum speed.

3. Draw an approximate graph of Speed vs. Distance for the motion. Put speed on the vertical axis. Measurements and scale numbers on the axis are not required. Label the points of maximum and minimum speed.

4. Draw a similar graph of Speed vs. Time for the motion. Put speed on the vertical axis.

5. In the diagram you drew for question (2), draw four velocity vectors for the object at different points along the path of motion. Show their direction and approximate relative sizes. Now draw vectors for the horizontal and vertical components of velocity in the picture at the same four points.

6. Draw a graph of Distance vs. Time for the motion. Put distance on the vertical axis. Make sure your qualitative graph is consistent with your speed vs. time graph in question 4.

7. a. List all other objects or substances that have an effect on the moving object. Which is the least important?
   b. In your drawing for question (2), show the direction of all forces acting on the moving object at four different points along the path of motion. Make longer force vectors for stronger forces and shorter force vectors for weaker forces. Label each vector with a letter and include a key which gives each force a name.

8. Draw in and label a separate vector showing the approximate directions and size of the total net force on the moving object at each of the four points.

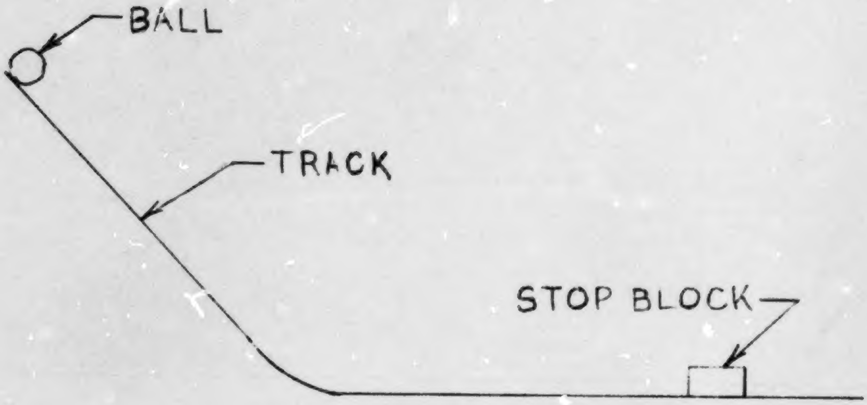
9. Write a paragraph describing how the speed of the object varies and why. You should refer to particular points on your graphs and/or diagrams that you have labelled with letters.

10. Write an abstract which states your major findings from the lab.

Qualitative Physics

Laboratory #2

Motion on the Flex-Track



Description: A ball rolls down an inclined section of track and collides with a block placed on the horizontal section.

Dependent Variables: Distance block moves
Velocity of ball

Possible Independent Variables:
- Height above horizontal surface of release point
- Mass of ball
- Size of ball
- Mass of block
- Distance traversed on angled portion of track
- Angle of track

Materials:
- Flex track
- Support grid and fasteners
- Assorted balls
- Meter stick
- Stop block
- Stopwatch

1. In column one below, list 3 or more factors you can change in the system, and in column 2 predict whether or not you think each will make a difference in what happens. Where you think changing a factor will make a difference, say carefully what it will affect, and whether it will increase or decrease the magnitude of that variable. You will be graded on how clearly you state your predictions, not on whether your predictions are correct. (In other words, feel free to make intuitive guesses!) Test each prediction and state what happened in column three.

(1) Factor Changed	(2) Predicted Effect(s)	(3) Observed Effect(s)

2. Using an entire page draw a schematic of the apparatus used in the experiment. Label the point(s) where the moving object is at maximum speed and where it is at minimum speed.

3. Draw an approximate graph of Speed vs. Distance for the motion. Put speed on the vertical axis. Measurements and scale numbers on the axis are not required. Label the points of maximum and minimum speed.

4. Draw a similar graph of Speed vs. Time for the motion. Put speed on the vertical axis.

5. In the diagram you drew for question (2), draw four velocity vectors for the object at different points along the path of motion. Show their direction and approximate relative sizes. Now draw vectors for the horizontal and vertical components of velocity in the picture at the same four points.

6. Draw a graph of Distance vs. Time for the motion. Put distance on the vertical axis. Make sure your qualitative graph is consistent with your speed vs. time graph in question 4.

7. a. List all other objects or substances that have an effect on the moving object. Which is the least important?
   b. In your drawing for question (2), show the direction of all forces acting on the moving object at four different points along the path of motion. Make longer force vectors for stronger forces and shorter force vectors for weaker forces. Label each vector with a letter and include a key which gives each force a name.

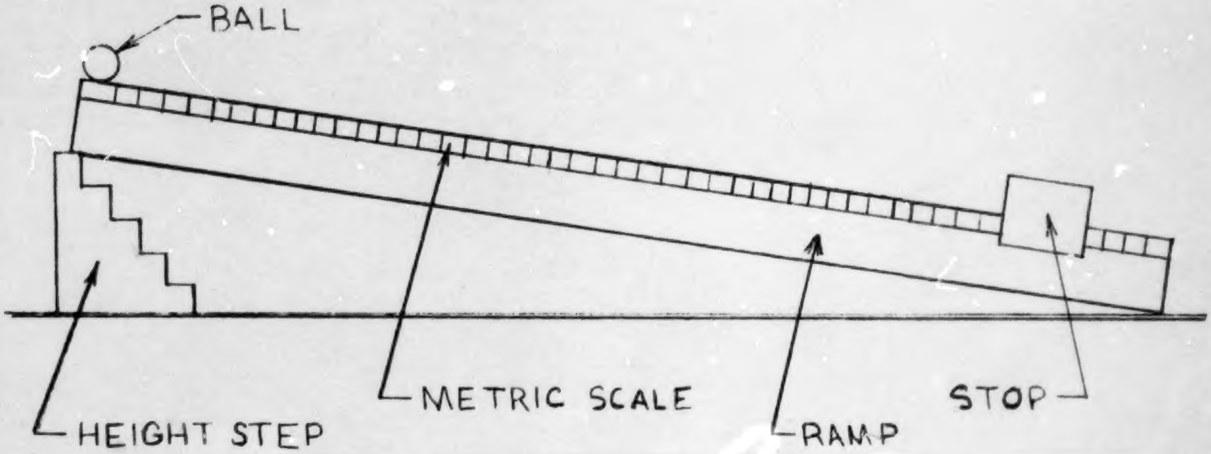
8. Draw in and label a separate vector showing the approximate directions and size of the total net force on the moving object at each of the four points.

9. Write a paragraph describing how the speed of the object varies and why. You should refer to particular points on your graphs and/or diagrams that you have labelled with letters.

10. Write an abstract which states your major findings from the lab.

Qualitative Physics
Laboratory #3

Acceleration on the Inclined Plane



Description: Ball is released from various positions along ramp. Position at one second time intervals is marked, and acceleration is analyzed qualitatively.

Dependent Variables:
- Veclocity of ball(s)
- Acceleration rate of ball(s)
- Time to traverse ramp

Possible Independent Variables:
- Angle of ramp
- Height from horizontal
- Distance traversed on ramp
- Mass and size of ball(s)

Materials:
- Wood ramp
- Assorted balls
- Notched height step
- Metronome
- Ball stop block
- Meter stick
- Markers

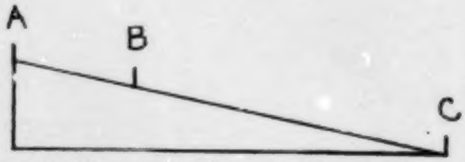
Notes: Attached prediction sheet is given as a quiz (not for credit) and collected prior to beginning lab. Students are then asked to test and verify their predictions. This is done in lieu of question 1. This laboratory overlaps with laboratory 2 to some extent. However, depending on the average level of the students, many or all of them will benefit from this redundancy. Students were also asked to draw a graph of acceleration vs time instead of speed vs time for question 4.

Name________________________

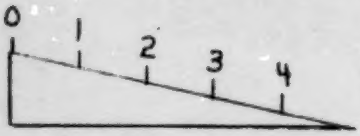
INCLINED PLANE LABORATORY

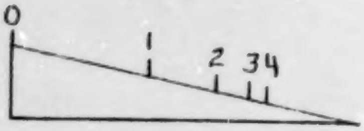
Give a one or two sentence reason on a separate sheet for each answer.

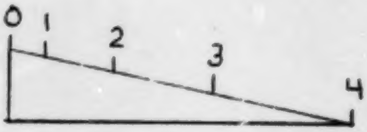
1) PREDICTION: The same ball is released first at A then at B, stopping at C each time. Which takes longer?

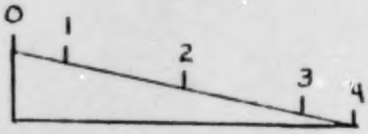


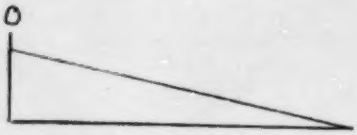
2) PREDICTION: If you raise the ramp 3" and mark the position of the ball with chalk at 1 second intervals, circle what it will look like:

a)


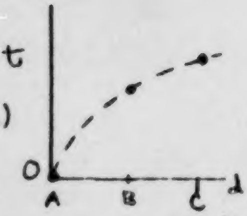
b)


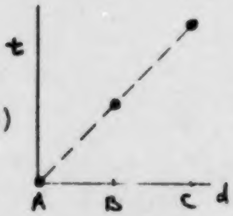
c)


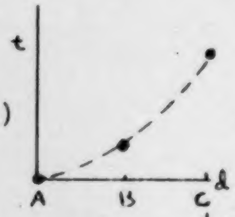
d)


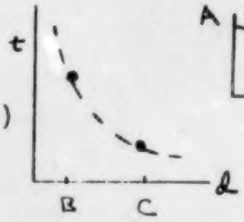
Other


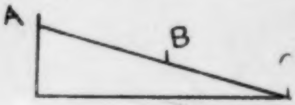
3) PREDICTION: If you graph the time it takes for the ball to go from A to B and A to C in the drawing below, it will look like

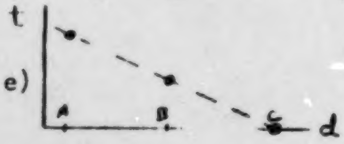
a)


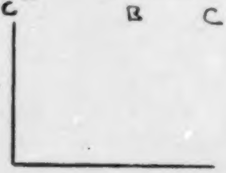
b)


c)


d)




e)


Other


2. Using an entire page draw a schematic of the apparatus used in the experiment. Label the point(s) where the moving object is at maximum speed and where it is at minimum speed.

3. Draw an approximate graph of Speed vs. Distance for the motion. Put speed on the vertical axis. Measurements and scale numbers on the axis are not required. Label the points of maximum and minimum speed.

4. Draw a similar graph of Speed vs. Time for the motion. Put speed on the vertical axis.

5. In the diagram you drew for question (2), draw four velocity vectors for the object at different points along the path of motion. Show their direction and approximate relative sizes. Now draw vectors for the horizontal and vertical components of velocity in the picture at the same four points.

6. Draw a graph of Distance vs. Time for the motion. Put distance on the vertical axis. Make sure your qualitative graph is consistent with your speed vs. time graph in question 4.

7. a. List all other objects or substances that have an effect on the moving object. Which is the least important?
   b. In your drawing for question (2), show the direction of all forces acting on the moving object at four different points along the path of motion. Make longer force vectors for stronger forces and shorter force vectors for weaker forces. Label each vector with a letter and include a key which gives each force a name.

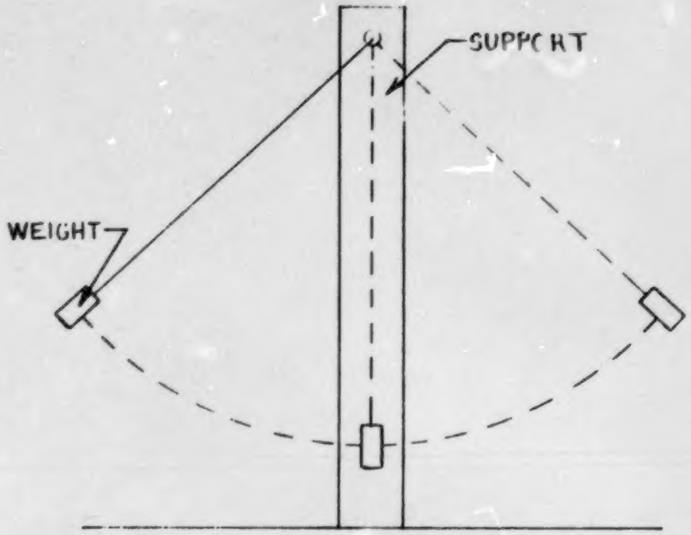
8. Draw in and label a separate vector showing the approximate directions and size of the total net force on the moving object at each of the four points.

9. Write a paragraph describing how the speed of the object varies and why. You should refer to particular points on your graphs and/or diagrams that you have labelled with letters.

10. Write an abstract which states your major findings from the lab.

Qualitative Physics
Laboratory #4

Harmonic Motion in the Pendulum



Description: The pendulum consists of assorted hook weights and a string of adjustable length. Students determine the factors which affect the period of the motion.

Dependent Variables: Period of the motion

Possible Independent Variables:
- Mass of hook weight
- Length of string
- Angle from vertical of release point
- Height from horizontal of release

Materials:
- 90° table clamp
- 3 ft and 1 ft rod or pendulum support
- 90° rod clamp
- Hook weight set
- String and clamp
- Stopwatch
- 2 meter sticks

1. In column one below, list 3 or more factors you can change in the system, and in column 2 predict whether or not you think each will make a difference in what happens. Where you think changing a factor will make a difference, say carefully what it will affect, and whether it will increase or decrease the magnitude of that variable. You will be graded on how clearly you state your predictions, not on whether your predictions are correct. (In other words, feel free to make intuitive guesses!) Test each prediction and state what happened in column three.

(1) Factor Changed	(2) Predicted Effect(s)	(3) Observed Effect(s)

2. Using an entire page draw a schematic of the apparatus used in the experiment. Label the point(s) where the moving object is at maximum speed and where it is at minimum speed.

3. Draw an approximate graph of Speed vs. Distance for the motion. Put speed on the vertical axis. Measurements and scale numbers on the axis are not required. Label the points of maximum and minimum speed.

4. Draw a similar graph of Speed vs. Time for the motion. Put speed on the vertical axis.

5. In the diagram you drew for question (2), draw four velocity vectors for the object at different points along the path of motion. Show their direction and approximate relative sizes. Now draw vectors for the horizontal and vertical components of velocity in the picture at the same four points.

6. Draw a graph of Distance vs. Time for the motion. Put distance on the vertical axis. Make sure your qualitative graph is consistent with your speed vs. time graph in question 4.

7. a. List all other objects or substances that have an effect on the moving object. Which is the least important?
   b. In your drawing for question (2), show the direction of all forces acting on the moving object at four different points along the path of motion. Make longer force vectors for stronger forces and shorter force vectors for weaker forces. Label each vector with a letter and include a key which gives each force a name.

8. Draw in and label a separate vector showing the approximate directions and size of the total net force on the moving object at each of the four points.

9. Write a paragraph describing how the speed of the object varies and why. You should refer to particular points on your graphs and/or diagrams that you have labelled with letters.

10. Write an abstract which states your major findings from the lab.

THE CONCEPT OF FUNCTION

Laboratory #5

Behavior of Springs

Description: Relationship between spring's displacement and the mass of the attached weights is analyzed.

Dependent Variables: Spring's displacement

Possible Independent Variables:
- Mass of Weight
- Length of Springs
- Strength of Springs
- Combinations of Springs
  (connecting identical springs)

Materials:
- Spring Stand with parallax viewing mirror.
- Slotted Weight set
- Assorted Springs

The Concept of Function

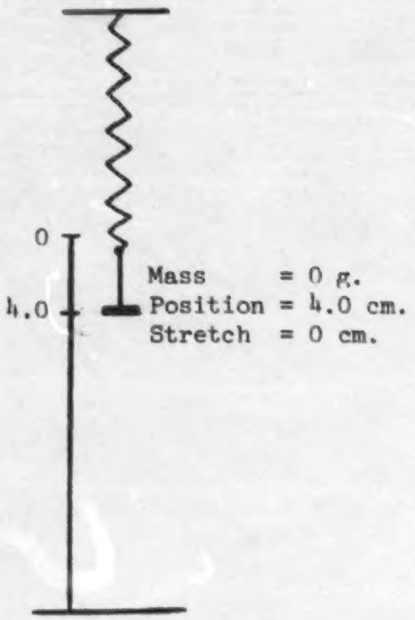
Laboratory #5

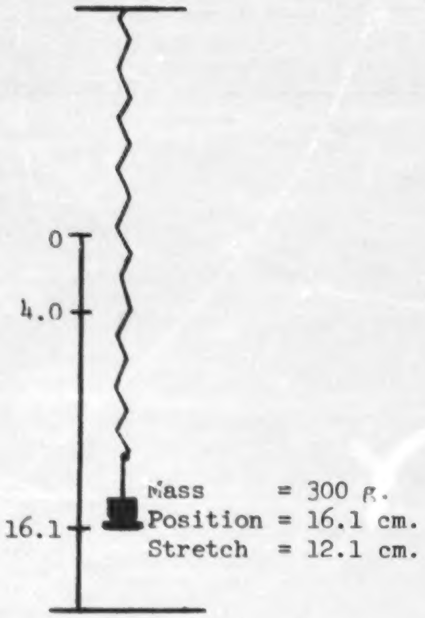
Behavior of Springs

Name ______________________

Date Turned-in ____________

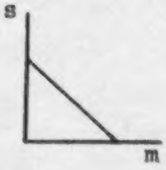
Measurement example:


0
4.0
Mass = 0 g.
Position = 4.0 cm.
Stretch = 0 cm.

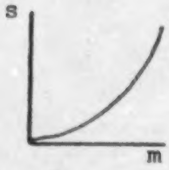

0
4.0
16.1
Mass = 300 g.
Position = 16.1 cm.
Stretch = 12.1 cm.

Equipment: Springs, stand, and set of slotted weights.

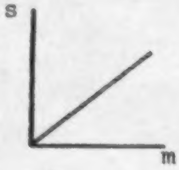
1) Predict the shape of a graph for a single spring where s is the amount the spring stretches from its original position, and m is the amount of mass placed on the holder at the bottom of the spring.



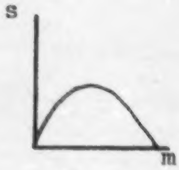
A



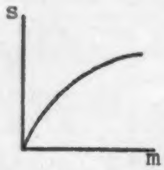
B



C



D



E

2) For a single spring:

a) Take data on 5 weights between 100g and 500 g. and make a graph. The data point with just the weight holder on the spring should be taken as $m = 0$, $s = 0$.

Equilibrium position with no weight on holder ____________.

Mass added to holder	$m =$	0	____	____	____	____	____
Position (Scale reading)	$p =$	______	____	____	____	____	____
Amount spring stretches from original position where $m = 0$	$s =$	0	____	____	____	____	____

Does the graph shape agree with your prediction in 1)? If not explain the difference to yourself.

b) Each person should use the graph to predict s for a new value of mass and write it here (don't show this prediction to your partner!)

	m	______
Predicted	s	______
Measured	s	______
% error		______

Have your partner check your predictions on the spring.

Calculate % error of prediction. (If your error is greater than 5%, you should recheck your data for question 2.)

3) Write an equation for the same spring which will predict the stretch for any weight between 0 - 500 g. Use s and m as two of your variables. Include the numerical value of any constants.

4) Use your equation to predict s for 242 g, and 358 g.

	m	242 g	358 g
Predicted	s		
Measured	s		

Calculate % error for your worst estimate.

If your error is more than 5% try to find the reason and improve your equation from question 3.

5) Use your equation or your graph to predict the mass needed to stretch the spring.

	s	12 cm	24 cm
Predicted	m		
Measured	m		

Have your partner check your predictions. Show your % error for your worst estimate.

If your error is more than 5%, try to find the reason and make a better prediction.

6) a) Copy your graph from question 2) using a dotted line. Draw in the predicted position and shape of graph for a stiffer spring (predict qualitatively here not quantitatively). Will this graph be above/below your first graph? Will it go through the origin? _________.
Check your prediction by graphing 3 data points. for a stiffer spring.

Mass	m			
Position	p			
Stretch	s			

b) Explain why the graphs for the two springs are different in the observed way, explaining why a difference in stiffness produces the difference seen in the graphs.

7) a) Write an equation for the stiffer spring using the variables s and m. Include the numerical value of any constants. Check your equation by making a prediction from it.

b) Compare this equation to the equation in question 3). In what way is the quantitative behavior of the two springs alike? Different?

8) a) Application questions: Springs behave in similar ways under both extension and compression. When a car is jacked up the wheels drop down some out of the wheel wells. How far down does a car settle on its rear wheels when k = 250 lbs/in. for each spring and the car weighs 3000 lbs? Assume rear wheels support 1/2 of the weight.

b) When the springs are designed, they must be made longer than they will be when installed. How much longer?

EXTRA CREDIT

9) a) Copy your graph from question 6. Make a quantitative prediction of the exact graph for two identical springs hooked together end to end of the kind you used in question 6. Show two predicted data points on your predicted graph.

b) Also predict an equation for this system relating m and s.

c) Check your graph and your equation against 2 data points using the spring.

d) Give a theory for why the equation for one spring is related to the equation for two springs in the way observed.

The Concept of Function:

Laboratory #6

Static Forces on the Inclined Plane

Description: Students determine quantitative factors which determine whether the system is balanced.

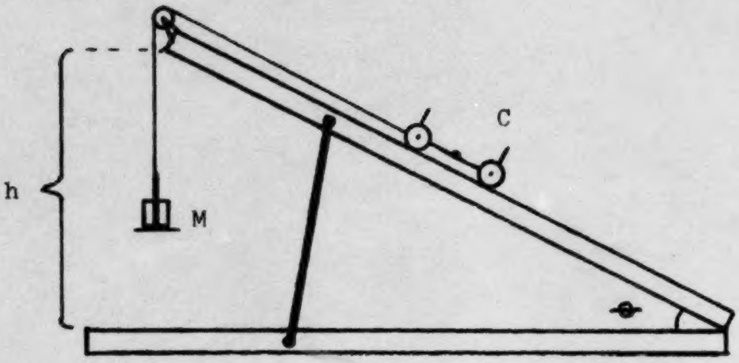
Variables: Height of end of ramp
Mass of cart and weights in cart
Mass hung on string

Materials: Slotted weight
Hooked weight set
Adjustable ramp
Pulley with clamp
String
1 pan balance, for group

Name ____________

Date Turned-in ________

The Concept of Function
Laboratory #6
Static Forces on the Inclined Plane



h = Height of plane

M = Mass hung on string

C = Mass of cart and weights placed in cart

1. a) If C is increased, what must be done to M to keep the system balanced?

Prediction:	__increase M __decrease M	Test Result:	__increase M __decrease M

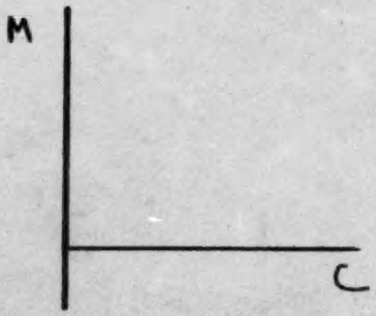
b) If C is increased, what must be done to h to keep the system in balance?

Prediction:	Test Result:

c) If h is decreased, what must be done to M to keep the system balanced?

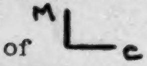
Prediction:	Test Results:

2) Predict the shape of the graph of M vs. C that shows how much mass must be put on the string to balance the mass of the cart, C. (M should include mass of weight holder). (C should include the weight of the cart).



For what values of C will the slope be less than 1 ?

For what values of C will the slope be greater than 1?

3. a) Take the data on 4 values of C between 0 and 400g. and make a graph of  that predicts all the values where the cart will balance. Show your data table next to your graph.

Does the graph shape agree with your prediction in (2)? If not, explain the difference to yourself. The slope of the graph is:

___greater than one

___less than one

b) Each person should choose one new value for C and use the graph to predict the new value of M. Have your partner check your prediction

	C______
Predicted	M______
Measured	M______
% Error	______

If your error is greater than 5%, try to improve your calculation and measurement procedures.

4. a) Write an equation which will predict M for any given C (with the ramp in its present position).

b) Use your equation to predict M for C = 460g. (C = mass of the cart and the load in the cart).

C = ____

Predicted M = ____

Measured M = ____

% error ____

Try to improve your estimation process if your error is greater than 5%

4. c) Explain why your equation makes sense in terms of the apparatus. Does it show that M must be increased if C is increased? Does this make sense in terms of how the apparatus is set up? Is the constant in the equation greater or less than one? Does this make sense in terms of the apparatus?

5) You're on your own on this one!
Experiment with the equipment to find an equation which will predict M, the mass needed to balance the cart in terms of h, the height to which the end of the board is raised. Keep the cart empty during your experiments. Show a data table with 3 pairs (M,h) next to your equation. Check your equation by predicting and testing a fourth pair of values, showing your % error. (See the drawing on page one for the exact method of measuring h.)

6) Find a single "Master Equation" which predicts values for M, given any values of h and C. (Express M as a function of h and C.) Test your equation at at least two data points.

h		
C		
Predicted M		
Measured M		

Is your equation consistent with your findings in questions

(1.a) ?

(1.b) ?

(1.c) ?

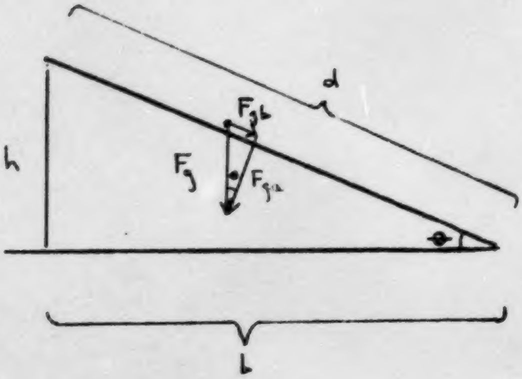
7) In the picture on p. 1 draw in vectors showing all forces acting on the empty cart when it is balanced. Provide a key, giving a name for each force.

EXTRA CREDIT

8) Derive the Master Equation for M by analyzing the system.

HINT: In the drawing below, the gravity force vector, F_g has been split into two perpendicular components, F_{ga} and F_{gb}. If they were present as forces, they would have the same effect on the cart as F_g alone does. F_{ga} is counteracted by a normal force from the ramp, leaving F_{gb} as the force that plays a role in balancing the cart. Notice that the triangle formed by the force vectors is similar to the triangle with sides d, h, and b. Also notice that the forces F_{gb} and F_g can be rewritten in terms of other variables.

Measure d and compare your derived equation with your equation in (6) to see how closely they match.



THE CONCEPT OF FUNCTION

Laboratory #7

Acceleration on the Air Track

Description: Quantitative analysis of the motion of an object undergoing constant acceleration.

Dependent Variables: Velocity of cart.
Acceleration rate of cart.

Independent Variable: Angle of track from horizontal.

Materials: Airtrack with Vacuum blowers
Air cars
Weight set
Spark generator, tape, etc.
Pan balance
Extra long meter sticks
1/8" Metal tiles for raising track end

The Concept of Function: Acceleration on the Air Track
Laboratory #7

Name: ____________________

Date: ________________

1) Level your air track. With the track level, make a run using the spark tape. Calculate the speed of the run by measuring the distance between a pair of adjacent dots. Measure several pairs and take the average.

Slow run

average distance between dots ________

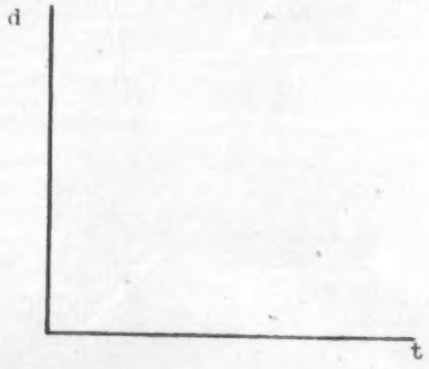
time interval between dots ________

speed (cm/sec) ________

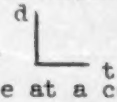
Watch again to "eyeball" how many centimeters the sled seems to cover in 2 seconds. Does this make your speed calculations above seem reasonable? For a faster run, will the marks be closer or farther apart?

2a) Raise one end of the track so that the sled traverses the track in 6 seconds or less. Record the height, and predict the qualitative shape of a graph of total distance traveled along the track vs. time. Use cm. and seconds as units.

height = ________



2b) Take the data on values of d after each second (every 10 dots = 1 sec.)

Make a graph of  that predicts all the values d where the sled will be at a certain time. Show your data table next to your graph.

Does the graph shape agree with your prediction in (2)? If not, explain the difference to yourself.

2c) Each person should choose one new value for t and use the graph to predict the new value of d. Have your partner check your prediction on the tape.

t________

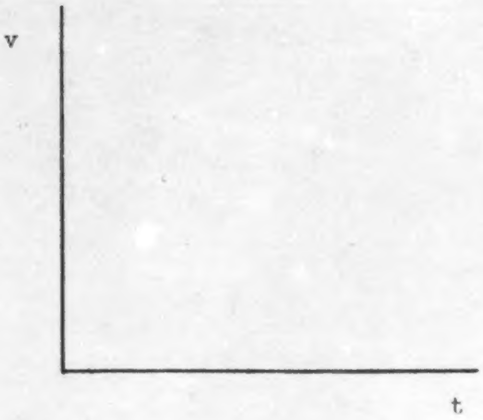
Predicted d________

Measured d________

% error ________

If your error is large, try to improve your calculation and measurement procedures.

3a) In question one we found that we can measure the velocity at any point by measuring the distance between two adjacent marks and relating it to the time interval between the marks. Predict the shape of a graph of velocity vs. time for the same tape you used in question 2.



t	cm/.1sec	cm/sec
_____	_____	_____
_____	_____	_____
_____	_____	_____
_____	_____	_____
_____	_____	_____
_____	_____	_____

b) Measure the distance between marks at 1 sec. intervals to plot the actual graph. Use the data table above.

c) Predict a new data point, test it, and give % error.

d) Write an equation for V in terms of t for your data. Include numerical values for constants.

e) What is the slope of your graph? ______

What is the acceleration of the sled? _____ (include units)

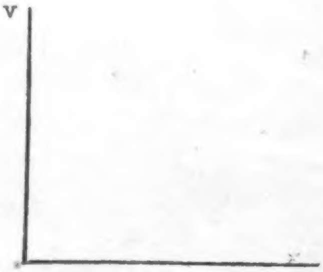
4a) Assume that the sled's acceleration will be proportional to h, the height you raise the end of the track. Pick a new value for h and predict the new value of acceleration. Check your prediction by making a new tape.

Old value h= _____

New value h= _____

Old value a= _____

Predicted new value a= _____



Predicted graph
(show how it will differ from graph in ques. 3)

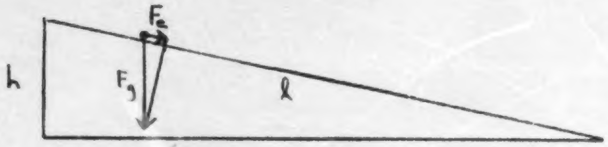
b) Make a new data table and graph of v vs t for the new height.

c) Write an equation for V in terms of t

d) Slope of graph = _____

New acceleration = _____ Predicted acceleration (4a) ________ % error ____

5) Calculate the sled's acceleration theoretically for one of the heights you used, and compare the calculated value with your measured value in 4.



$F_g = mg$

Do this using the similar triangles shown above. Assume $g = 980 \text{ cm/sec}^2$ = acceleration due to gravity downward. Using your value of h, determine the value of F_e, the component of gravitational force working along the line of motion, in terms of m and g. Then use F_e to determine the acceleration of the sled with mass m.

h = _____

actual a = _____

calculated a = _____

6) Write a paragraph on why the distance vs. time graph is curved. You should relate this to: 1. The distance between marks on the tape; 2. the acceleration of the sled; 3. the instantaneous speed of the sled.